ATOC 7500: The Art of Climate and Environmental Modeling

Week 7 Lab Assignment

Weather prediction with a barotropic vorticity model

To a good approximation atmospheric flow can be considered mostly horizontal and geostrophic. That is to say the flow is parallel to geopotential height contours. Given this, the horizontal momentum equations can be written terms of a single variable, relative vorticity:

$$z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

We know from conservation of angular momentum arguments that the absolute vorticity ($h = z + f$) is conserved,

$$\frac{dh}{dt} = 0$$

so we can write an evolution equation for our state variable:

$$\frac{\partial z}{\partial t} = -u\frac{\partial}{\partial x}(z + f) - v\frac{\partial}{\partial y}(z + f)$$

where the Coriolis parameter $f = 2\Omega \sin q \approx f_0 + by$ is the local vorticity associated with the Earth rotation. Our prediction equation thus states the vorticity change is due only to advection of absolute vorticity. Making use of the fact the flow is non-divergent we can rewrite this to a more convenient form:

$$\frac{\partial z}{\partial t} = -\frac{\partial}{\partial x}[u(z + f)] - \frac{\partial}{\partial y}[v(z + f)] \qquad (1)$$

$$\frac{\partial z}{\partial t} = -\frac{\partial}{\partial x}[F] - \frac{\partial}{\partial y}[G]$$

With absolute vorticity fluxes $F = u(z + f)$ and $G = v(z + f)$. Recall the vorticity and u and v wind are all functions of the stream function:

$$z = \nabla^2 y \qquad (2)$$

$$u = -\frac{\partial y}{\partial y} \quad \text{and} \quad v = \frac{\partial y}{\partial x} \qquad (3)$$

Writing a finite difference analog of (1) we have (assuming hereafter Δx = Δy)

$$z_{i,j}^{n+1} = z_{i,j}^{n-1} + 2\Delta t \left(\frac{\partial z}{\partial t} \right)_{i,j}^{n} \qquad (4)$$

$$\left(\frac{\partial z}{\partial t} \right)_{i,j}^{n} = -\frac{\left[F_{i+1,j}^{n} - F_{i-1,j}^{n} \right]}{2\Delta x} - \frac{\left[G_{i,j+1}^{n} - G_{i,j-1}^{n} \right]}{2\Delta x} \qquad (5)$$

The challenge remains to find the streamfunction from the known vorticity. Given (2), this equates to inverting the Laplacian. That is, solving Poisson's equation. Being elliptical we know this is possible given two appropriate boundary conditions.

A solution can be found using a so-called relaxation method to iteratively find the solution. Consider the finite difference analog of (2).

$$z_{i,j} = \frac{1}{(\Delta x)^2} \left(y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1} - 4y_{i,j} \right) \qquad (6)$$

Rearranging, we find,

$$y_{i,j} = \frac{\left(y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1} \right)}{4} - \frac{(\Delta x)^2}{4} z_{i,j} \qquad (7)$$

It can be shown that given a known vorticity (and the boundary conditions for the streamfunction) a the $y_{i,j}$ can be found from a guess at the surrounding values. That is for some number of iterations m.

$$y_{i,j}^{m+1} = \frac{\left(y_{i-1,j}^{m} + y_{i+1,j}^{m} + y_{i,j-1}^{m} + y_{i,j+1}^{m} \right)}{4} - \frac{(\Delta x)^2}{4} z_{i,j} \qquad (8)$$

Iteration is stopped once the solution converges to the required accuracy. We take the boundary conditions that stream function on the north and south boundaries as uniform and the zonal mean (average over all point *i*) of the initial conditions (i.e., $y_{i,1}^{n} = \overline{y_{1}^{0}}$ and $y_{i,nlat}^{n} = \overline{y_{nlat}^{0}}$). Further we assume the domain is periodic in the longitudes *x*.

Optional part: We can refine our iterative solver by noting that this scheme exponentially approaches the true solution. This can be accelerated by "over relaxation"

$$y_{i,j}^{m+1} = s \left\{ \frac{\left(y_{i-1,j}^{m} + y_{i+1,j}^{m} + y_{i,j-1}^{m} + y_{i,j+1}^{m} \right)}{4} - \frac{(\Delta x)^2}{4} z_{i,j} \right\} + (1 - s) y_{i,j}^{m} \qquad (9)$$

where σ is an over relaxation parameter that is between 1 and 2 (values of 1.8 work well). This can be seen as an extrapolation of the iteration to get a better guess, and roughly halves the number of iterations needed.

We can now construct a numerical weather prediction model. Given the initial geopotential we can compute initial streamfunction ($y = g\Phi / f_0$), and thus compute initial vorticity. This is the initial conditions. The model can be characterized by the following steps:

1) Solve Poisson's equation iteratively using (8) to obtain the y from *?*
2) Compute the finite gradients of y in *y* and *x* to find *u* and *v*
3) Construct fluxes F and G
4) Compute the finite gradients and *x* and *y* of *F* and *G* to evaluate the time derivative
5) Perform centered time step (with filtering), and update for next step

… continue looping…

Given that first derivatives in *x* and *y* are straight forward with finite differences, the challenge in this model is step 1, solving the Poisson equation.

Choose a grid of nlon=64 and nlat=16, with $\Delta x = \Delta y = 500$ km grid spacing and a time step of 1 hour. Typical mid-latitude values for the Coriolis parameter can be deduced from $f_0 = 1\text{x}10^{-4}\ s^{-1}$ and $\beta \sim 1\text{x}10^{-11}\ s^{-1}m^{-1}$. Given the observed vorticity, make a 5 day weather prediction.

Tips for implementation

- Start by writing the finite difference code (subroutines) to compute *x* and *y* derivatives and use these to evaluate the u and v wind components from streamfunction.
- Write code to evaluate the vorticity fluxes, and use your finite difference subroutines to evaluate the advective time derivative
- A confirmation of your Poisson solver will be to start from a known streamfunction compute the laplacian to get vorticity, then invert the Laplacian to get the original streamfunction
- Initially you will want output every time step to make sure the model works. After that, consider writing output every 6 hours to save disk space.
- You may like to output vorticity, u, v and geopotential for comparison with observed weather.
- Such a model can sustain Rossby waves of the form before trying a weather prediction, convince yourself that the model correctly simulates single Rossby waves:

  ?=sin(kx)cos(ly)

  (be careful to ensure the vorticity is zero on the north and south boundaries).

Discussion questions

- Does your model correctly simulate Rossby waves?
- Does your model simulate all wave number just as well?
- How accurate is your prediction? What features are well modeled? What features are missed?
- Does your solution remains stable for long simulations?
- How does the addition of surface drag effect results?
  (add surface friction as -D? with D a dissipation rate of about $0.1\text{x}10^{-5}\ s^{-1}$)
- How does horizontal diffusion affect results?
  (add horizontal diffusion as $K\nabla^2$? with K an eddy diffusivity of $2.5\text{x}10^{-5}\ m^2s^{-1}$)