ATOC 7500: The Art of Climate and Environmental Modeling
Week 6 Lab Assignment
Soil heat diffusion

In Boulder the ground is frozen during winter, and plants can not grow in frozen soil. During spring, the soil thaws. When should I plant tomatoes? When will the tomatoes die (from the first frost)?

Let us define the time to plant as when the top 0.5 meters of soil is unfrozen, and construct a model to predict when this will occur.

We can write the heat flux at any depth in the soil as

$$F = -K\frac{\partial T}{\partial z}$$

Thus the temperature change can be found as the convergence of the flux (notice double negative signs). So

$$\frac{\partial T}{\partial t} = \frac{1}{C_p}\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)$$

Where C_p is the heat capacity of the soil (i.e., related to the specific heat and density $C_p = \rho c_p$). Assuming the diffusivity is constant with depth, we can write a finite difference analog (with $k = K/C_p$) for temperature on a grid z_j with grid uniform spacing Δz:

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} = k\left(\frac{T_{j-1}^{n+1} - 2T_j^{n+1} + T_{j+1}^{n+1}}{(\Delta z)^2}\right)$$

Here we have a scheme which is implicit because variables at time n+1 occur both in the time and space derivative. (If instead we evaluate the space derivative at time n we have an explicit, forward scheme, which is conditionally stable). The implicit scheme can be solved by constructing a matrix of simultaneous equations, with the numerical challenge being the inversion of the matrix methods. While generic methods are well known, we can write a system of equations represented by tridiagonal matrix, for which a solution can easily be found.

Heat is supplied to the soil by energy flux from the atmosphere which may be written as:

$$F_{surface} = D(T_{air} - T_1)$$

Where D is an exchange coefficient (taken as D = 1.157x10^{-5}/s), and T_{air} is the surface air temperature. This can be approximated as:

$$Tair = T_0 + A_a \cos(2\pi d/365)$$

where T_0 is the annual mean air temperature (280 K), A_a is the amplitude of the annual cycle (A_a = 20K), and d is the simulation day (0-365).

Assume the top 1 m of soil is sandy (k = 1.0x10^{-8} m^2/s).

Discussion questions

- How does the result change if I aim to plant on the other side of the yard where the soil is clay rich (k = 0.7×10^{-8} m^2/s)
- How does the result change if we include a diurnal cycle (A_d = 5K)?
  (Hint, think about thaw, AND the first frost!)
- What is the time mean soil temperature?
- Is the result the same for all time steps?
- Does the scheme conserve mass?
- Does the scheme conserve variance?
- Is the system positive definite?
- Is the phase and amplitude of the seasonal cycle the same at all depths?
- What is the sole of the surface heat flux (and the magnitude of D) in the result?
- Do your predictions match the date given by Farmer Almanac?

Possible useful graphs

- Temperature as a function of time for various depths
- Temperature as a function of depth for various times
- A contour plot of temperature as a function of depth and time