Boundary layers

Equations for the boundary layer

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} - f\bar{v} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{\partial}{\partial z}\left(\overline{u'w'}\right)$$

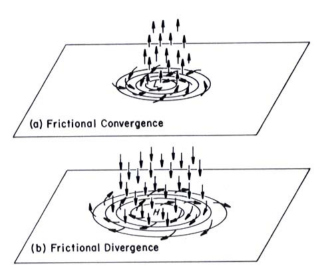
$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z} + f\bar{u} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - \frac{\partial}{\partial z}\left(\overline{v'w'}\right)$$

Simplify for balanced flow
(making use of geostrophic definition),

$$f\left(\bar{v} - \bar{v}_g\right) = \frac{\partial}{\partial z}\left(\overline{u'w'}\right)$$

$$f\left(\bar{u} - \bar{u}_g\right) = -\frac{\partial}{\partial z}\left(\overline{v'w'}\right)$$

States that turbulent transfer is responsible
for a geostrophic flow



Flux-Gradient Theory and the Mixing Length Hypothesis

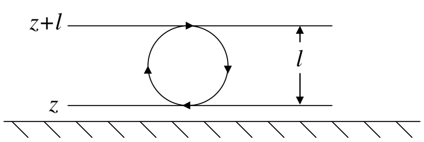
- A very simple form of turbulence closure can be obtained by assuming that a vertically displaced parcel of fluid carries its mean horizontal velocity to its new level, where it completely mixes with its new environment and causes a turbulent velocity fluctuation.

  i.e $\overline{u'w'} = -C_d |\mathbf{V}| \overline{u}$

- Another way, a single parameter, representing the typical scale of an eddy, is required. This parameter is known as the "mixing length."

Mixing length hypothesis

A vertically displaced parcel of fluid, originally at level z, carries its mean horizontal velocity $\overline{u}(z)$ to another level $z+l$. The mean velocity at $z+l$ is $\overline{u}(z+l)$
Assume that the parcel completely mixes with its environment at the new level, causing a turbulent fluctuation u'.



$$u' = -l\left[\frac{\overline{u}(z+l)-\overline{u}(z)}{l}\right] \cong -l\frac{\partial \overline{u}}{\partial z}$$

l = "mixing length"

Thus quantities in eddy stress terms, such as $-\rho\overline{u'w'}$ can be written as $-\rho\overline{w'l}\frac{\partial \overline{u}}{\partial z}$.

If we further assume that turbulence is about the same in all directions ("isotropic"), then:

$|w'| \approx |u'|, |v'| \rightarrow w' = l\left|\frac{\partial \overline{u}}{\partial z}\right|$ Thus $\rho\overline{u'w'} = \rho\overline{l^2}\left|\frac{\partial \overline{u}}{\partial z}\right|\frac{\partial \overline{u}}{\partial z}$ If we define $K_m = \overline{l^2}\left|\frac{\partial \overline{u}}{\partial z}\right|$ (eddy viscosity coefficient)

Then we can write the eddy stress terms as $-\rho\overline{u'w'} = \rho K_m \frac{\partial \overline{u}}{\partial z}$ and $-\rho\overline{v'w'} = \rho K_m \frac{\partial \overline{v}}{\partial z}$

Ekman layer

We can move toward a solution to these equations by using the definition of the geostrophic wind to substitute for the pressure gradient term.

$$-f\overline{v} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + K_m \frac{\partial^2 \overline{u}}{\partial z^2} \qquad v_g = \frac{1}{\rho f}\frac{\partial \overline{p}}{\partial x}$$

$$f\overline{u} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + K_m \frac{\partial^2 \overline{v}}{\partial z^2} \qquad u_g = -\frac{1}{\rho f}\frac{\partial \overline{p}}{\partial y}$$

After some simple manipulation, we obtain the classical Ekman layer equations (named for Swedish oceanographer V. W. Ekman):

$$K_m \frac{\partial^2 \overline{u}}{\partial z^2} + f\left(v - v_g\right) = 0$$

$$K_m \frac{\partial^2 \overline{v}}{\partial z^2} - f\left(u - u_g\right) = 0$$

These equations can be solved to determine the departure of the wind from geostrophic balance as a function of height in the boundary layer.

Flow in the Ekman layer

Ekman layer equations are second order differential equations. To solve establish some boundary conditions.

$$u = 0, \quad v = 0 \quad at \quad z = 0$$

$$u \to u_g, \quad v \to v_g \quad as \quad z \to \infty$$

With these boundary conditions, a can be obtained.

$$u = u_g\left(1 - e^{-\gamma z}\cos\gamma z\right)$$

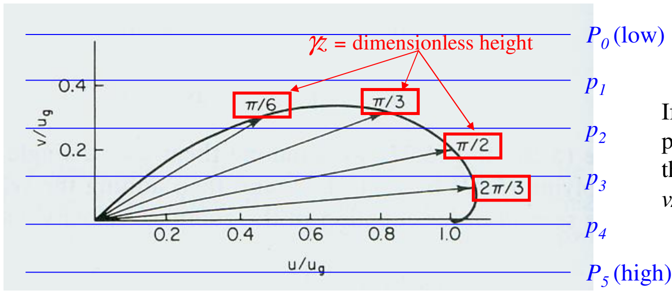
$$v = u_g e^{-\gamma z}\sin\gamma z$$

where $\gamma = \left(f / 2K_m\right)^{1/2}$

This solution, which is valid in the Northern Hemisphere, yields the Ekman spiral. The Ekman spiral describes the turning of the winds with height in the boundary layer as the effects of friction diminish with height.

Ekman spiral

The Ekman spiral depicts the turning of the wind with height.



If the wind was purely geostrophic, then $u/u_g = 1$ and $v/u_g = 0$.

•The wind approaches the geostrophic wind at the top of the PBL
•Within the PBL the wind blow across the isobars toward lower pressure.
• Cross-isobaric flow produces boundary layer convergence in cyclones and boundary layer divergence in anticyclones.

Boundary layer closure

Mixing length hypothesis

- Theoretical basis for selection K
- K smaller with stronger stability, larger with stronger wind shear
- In a uniform vertical gradient, a parcel lifted a height l, differs from the background by,

$$u' = \bar{u}_2 - \bar{u}_1 = \left(\bar{u} - l\frac{\partial \bar{u}}{\partial z}\right) - \bar{u} = -l\frac{\partial \bar{u}}{\partial z}$$

- Assuming spherical eddies $w' = l\left|\frac{\partial \mathbf{V}}{\partial z}\right|$

$$u'w' = -l^2\left|\frac{\partial \mathbf{V}}{\partial z}\right|\frac{\partial \bar{u}}{\partial z} = -K\frac{\partial \bar{u}}{\partial z} \qquad K = l^2\left|\frac{\partial \mathbf{V}}{\partial z}\right|$$

- *Includes information of eddy generation*
- *Requires selection of l* (~ 30 meters)

Surface layer

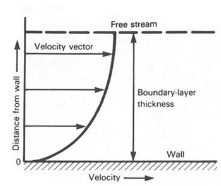
- Small enough that we can neglect Coriolis and pressure gradients (~10-30 m)
- Dependant only turbulent character (shear independent of height)
- Define a friction velocity
- Eddy size is limited distance to surface
  So $l = kz$
  (k is the von Karman constant ~ 0.3-0.4)

$$u_*^2 = \left(\overline{u'w'}\right)_{surface} = K\frac{\partial \overline{u}}{\partial z}$$

$$u_*^2 = (kz)^2\frac{\partial \overline{u}}{\partial z}$$

$$\overline{u} = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right)$$

- Integrating from u=0 at $z=z_0$
  (z_0 is the roughness length, depends on surface)



Wind speed increases logarithmically with height
Note $u_* = C_d u_{ref,}$ so we can estimate C_d by measuring the wind profile

Ekman layer

Using K-theory

$$f\left(v - v_g\right) = -K\frac{\partial^2 \overline{u}}{\partial z^2}$$

$$f\left(u - u_g\right) = K\frac{\partial^2 \overline{v}}{\partial z^2}$$

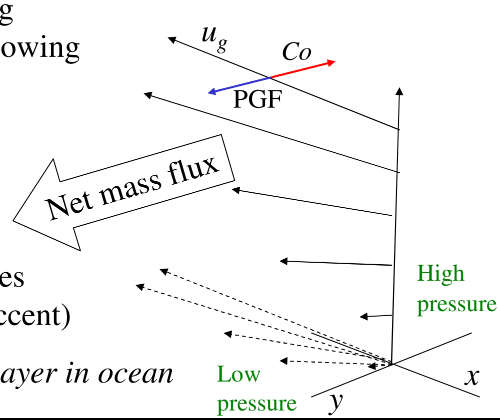
Ageostrophic acceleration balanced by turbulent momentum flux

Turbulence slows flow, reducing Coriolis closer to the surface allowing greater cross isobaric flow

Vertical integration shows net ageostrophic flow

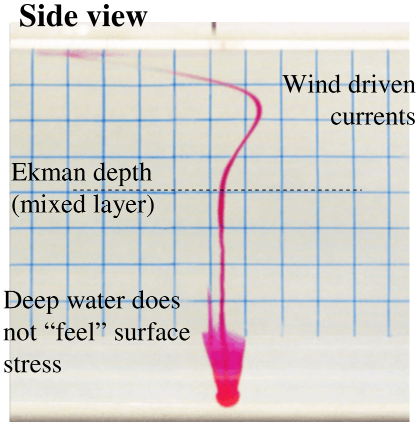
As such, convergence to cyclones (which must be matched with accent)

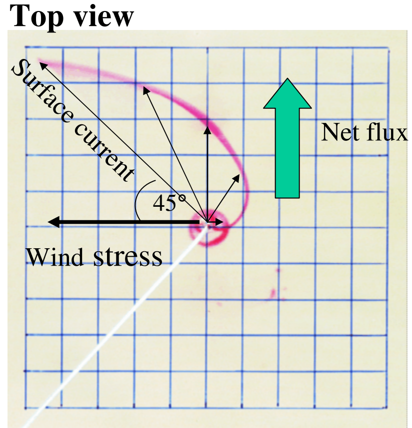
Differs somewhat from Ekman layer in ocean



Ekman layer in laboratory (ocean)

Rotating tank (Coriolis), with applied surface stress

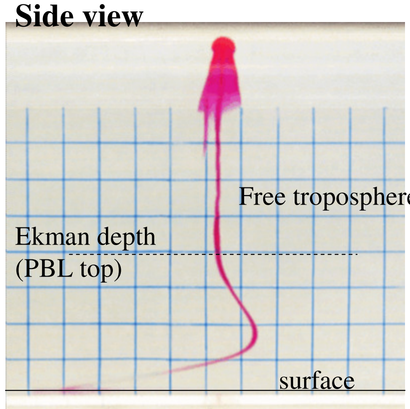


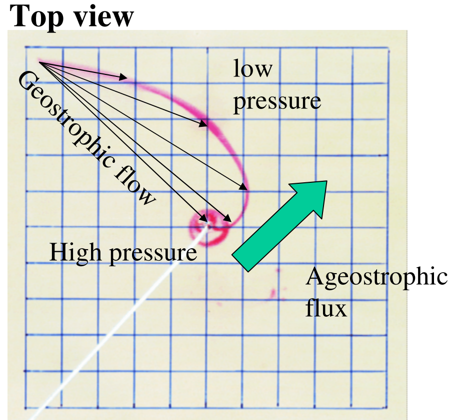


Seen in ocean. Ekman drift of sea ice (Ekman's observation), and upwelling where Ekman transport causes divergence

Ekman layer in laboratory (atmosphere)

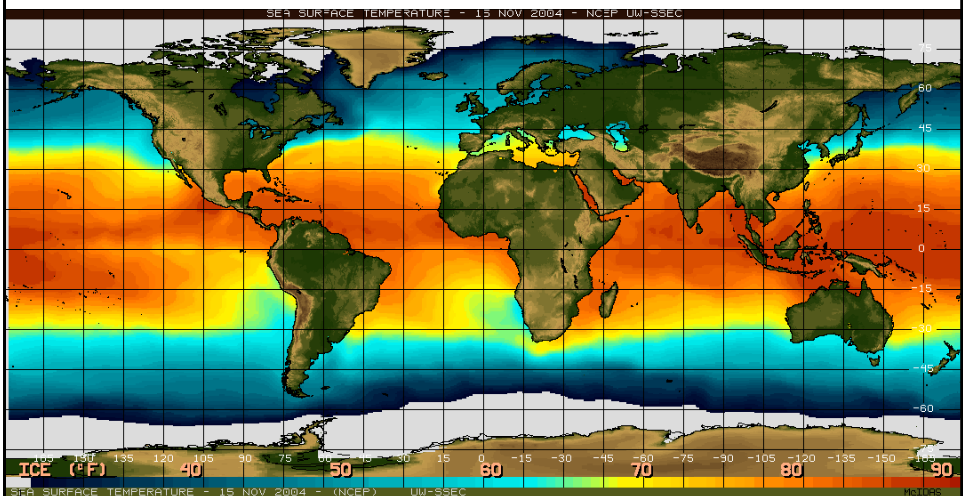
Rotating tank (Coriolis), with applied surface stress





Requires stable boundary layer – rarely seen easily in atmosphere
However, effects of turning are evident

Today's sea surface temperature



Boundary layer mass flux

- Decompose flow into "pressure driven" and "turbulent stress driven" components: $u=u_p+u_t,\ v=v_p+v_t$

$$\cancel{\frac{d\bar{u}_p}{dt}} = f\bar{v}_p - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} \qquad \cancel{\frac{d\bar{u}_t}{dt}} = f\bar{v}_t + K\frac{\partial^2 \bar{u}}{\partial z^2}$$

$$\cancel{\frac{d\bar{v}_p}{dt}} = -f\bar{u}_p - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} \qquad \cancel{\frac{d\bar{v}_t}{dt}} = -f\bar{u}_t + K\frac{\partial^2 \bar{v}}{\partial z^2}$$

•At steady state, assume geostrophic balance, and ageostrophic mass flow due to turbulence.

•Integrating over boundary layer depth to obtain cross isobaric (ageostrophic) mass flux

$$M_x = \int_0^d \rho_0 \bar{u}_t dz \qquad M_y = \int_0^d \rho_0 \bar{v}_t dz$$

Boundary layer divergence

- From continuity, horizontal mass flux divergence is associated with vertical motions.

$$w(d) - w(0) = -\frac{1}{\rho_0}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}\right)$$

0 0

Thus the Ekman pumping, $w(d)$
Alternatively, for balanced flow, can write,

$$w(d) = -\frac{1}{f}\left[\frac{\partial}{\partial y}\left(K\frac{\partial^2 \overline{u}_t}{\partial z^2}\right) - \frac{\partial}{\partial x}\left(K\frac{\partial^2 \overline{v}_t}{\partial z^2}\right)\right]$$

Ekman pumping and vorticity

Making use of the Ekman solution,

$$u = u_g\left(1 - e^{-\gamma z}\cos\gamma z\right)$$
$$v = u_g e^{-\gamma z}\sin\gamma z$$

$$\gamma^2 = f/2K$$

After some algebra, write Ekman pumping in terms of geostrophic vorticity,

$$w(d) = -\frac{1}{2\gamma}\left[\frac{\partial u_g}{\partial y} - \frac{\partial v_g}{\partial x}\right] = \zeta_g\left[\frac{K}{2f}\right]^{\frac{1}{2}}$$

Turbulent drag on rotating fluid causes vertical motion

Barotropic vorticity equation

$$\frac{d\zeta_g}{dt} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = f\frac{\partial w}{\partial z} \approx \frac{f}{H}w(d)$$

Since the thermal wind vanishes for barotropic conditions, integrate as before,

$$\frac{d\zeta_g}{dt} = -\left|\frac{fK}{2H^2}\right|^{\frac{1}{2}}\zeta_g$$

Differential, has exponential solution $\zeta_g(t) = \zeta_g(0)\exp(-t/\tau)$

Where the characteristic (e-folding) time is (typically about 4 days)

$$\tau = H\left|\frac{2}{fK}\right|^{\frac{1}{2}}$$

So in the absence of vorticity sources, weather features will almost completely dissipate in about 10 days (3 e-folds)

What does this tell us about weather prediction with barotropic vorticity equation?

Example: cup of tea

- A cup of tea is stirred and spoon is removed (i.e., put into cyclostrophic motion)
- The bottom friction slows down the spinning fluid in the cup near the bottom surface
- This breaks the balance between the PGF and centrifugal force.
- The stronger PGF causes the secondary circulation - flow towards the center of the cup near the bottom (pushing tea leaves to the center of the cup)
- Compensating flow above this layer away from the central axis over the remaining depth of the cup
- To conserve angular momentum, the spin has to slow down. This is analogous to the spin down of a cyclone.

Predictions based on tea leaves would best use vorticity equation

Circulation view of spin-down

- In the region of a cyclone, convergence in PBL causes vertical motion and divergence aloft in the free troposphere
- Consider a ring of parcels that divergence i.e., the area expands
- Circulation must be conserved, tangential velocity must continually decrease

$$\frac{dC}{dt} = -\oint \frac{1}{\rho} dp - 2\Omega \frac{d}{dt}(A \sin\phi)$$

- That is, outflow allows a Coriolis force which acts against the primary rotation

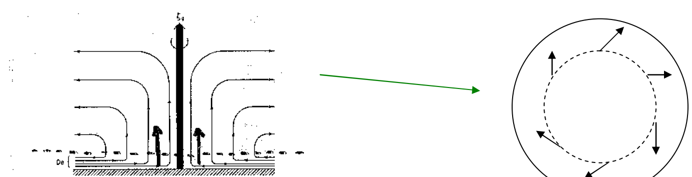


Fig. 5.7 Streamlines of the secondary circulation forced by frictional convergence in the planetary boundary layer for a cyclonic vortex in a barotropic atmosphere. The circulation extends throughout the full depth of the vortex.

Spin down experiment

Rotating tank. Geometry not quite the same as atmosphere.