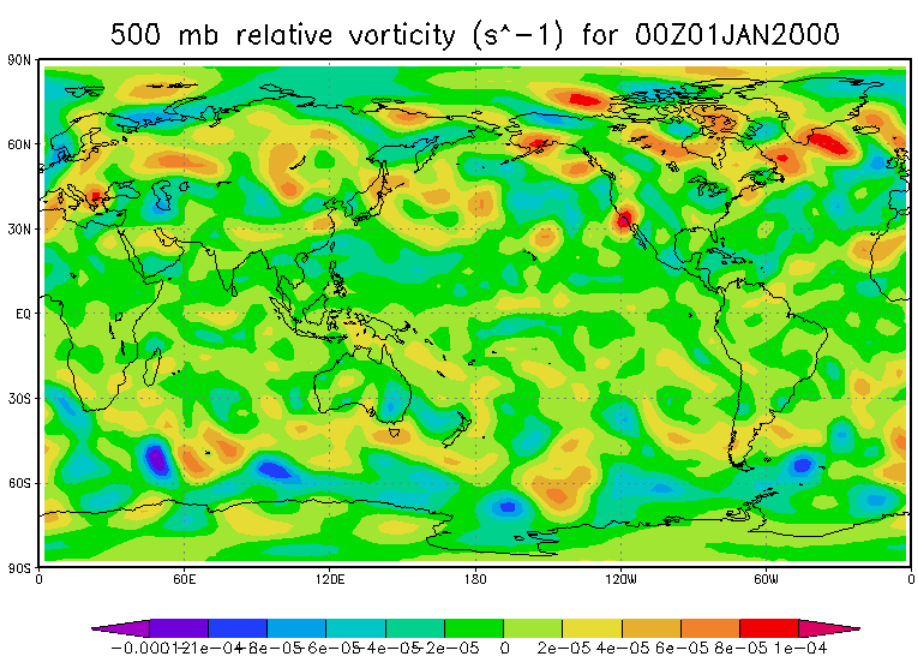
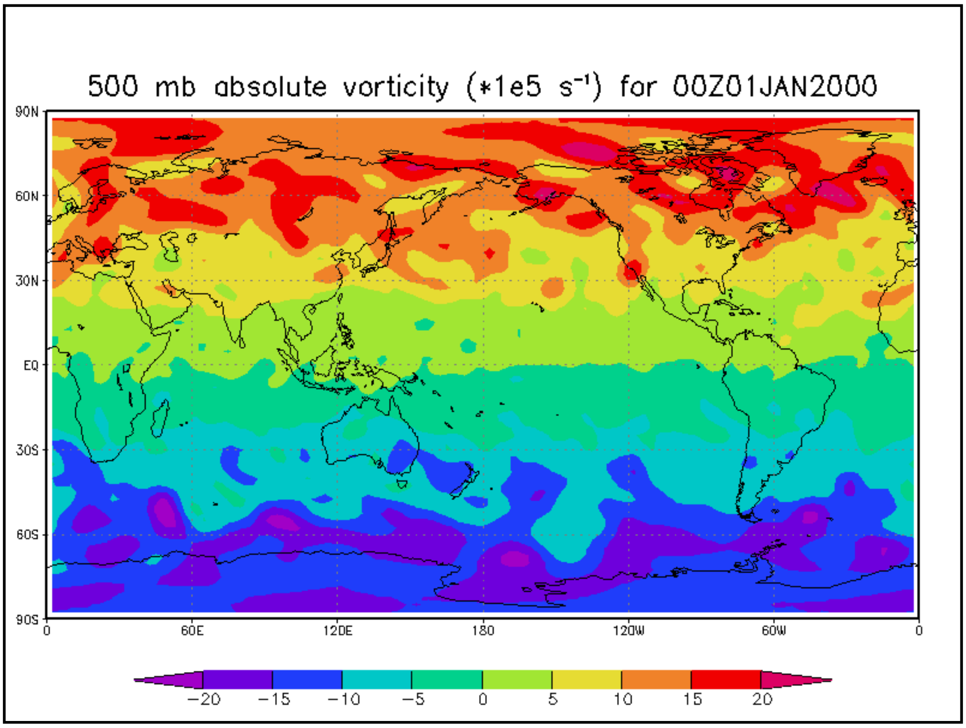
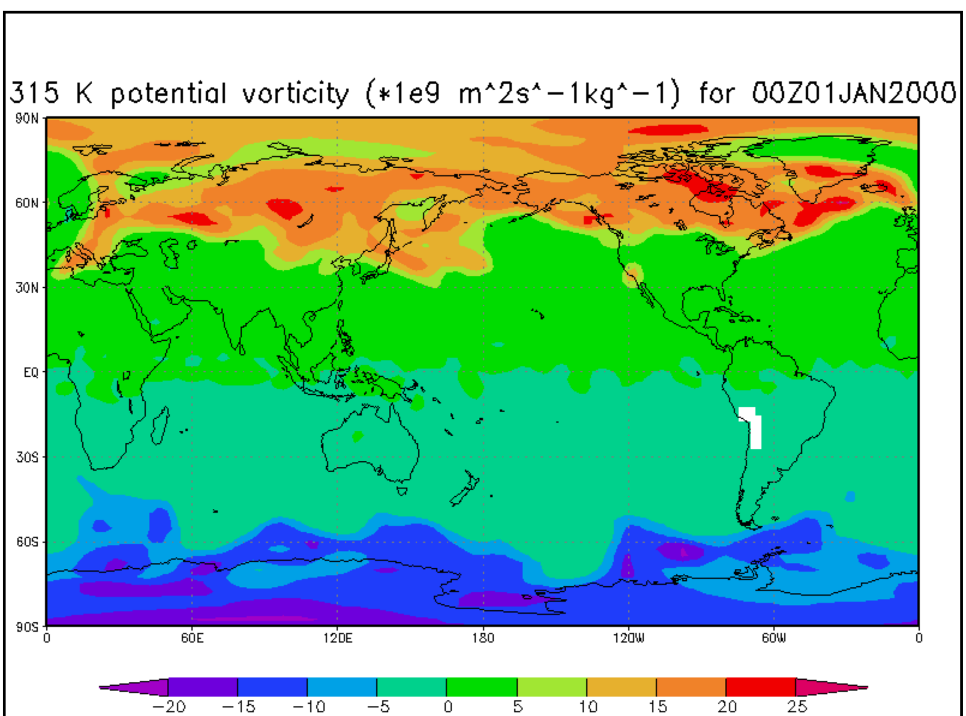
Vorticity equation 2

Why did Charney call it PV?


500 mb relative vorticity (s^-1) for 00Z01JAN2000
90N
60N
30N
EQ
30S
60S
90S
0
60E
120E
180
120W
60W
0
0
2e-05
4e-05
6e-05
8e-05
1e-04


500 mb absolute vorticity (*1e5 s^-1) for 00Z01JAN2000
90N
60N
30N
EQ
30S
60S
90S
0
60E
120E
180
120W
60W
0
−20 −15 −10 −5 0 5 10 15 20


315 K potential vorticity (*1e9 m^2s^−1kg^−1) for 00Z01JAN2000
90N
60N
30N
EQ
30S
60S
90S
0
60E
120E
180
120W
60W
0
−20 −15 −10 −5 0 5 10 15 20 25

The Vorticity Equation

Want to understand the processes that produce changes in vorticity.
So derive an expression that includes the time derivative of vorticity:

$$\frac{d\zeta}{dt} = \frac{d}{dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Recall that the momentum equations

$$\frac{du}{dt} = \text{Sum of forces in x direction}$$

$$\frac{dv}{dt} = \text{Sum of forces in y direction}$$

Thus we will begin our derivation by taking

$$\frac{d\zeta}{dt} = \frac{d}{dx}(\text{y momentum equation}) - \frac{d}{dy}(\text{x momentum equation})$$

Equivalently, we could use the vector form, and compute the curl of the vector momentum equation

$$\frac{d\zeta}{dt} = \mathbf{k} \cdot \left(\nabla \times \frac{d\mathbf{V}}{dt}\right)$$

Vorticity equation

We will work in Cartesian coordinates

The addition terms for spherical earth come out more naturally in vector form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{d\zeta}{dt}=\frac{d}{dx}(\text{y momentum equation})-\frac{d}{dy}(\text{x momentum equation})$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial x}\left[-\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)-\frac{1}{\rho}\frac{\partial p}{\partial y}+fu\right]$$
$$-\frac{\partial}{\partial y}\left[-\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)-\frac{1}{\rho}\frac{\partial p}{\partial x}-fv\right]$$

Vorticity equation (continued)

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial t}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+v\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}+w\frac{\partial^2 v}{\partial x\partial z}+\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}+f\frac{\partial u}{\partial x}+u\cancel{\frac{\partial f}{\partial x}}=-\cancel{\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}}+\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right)$$
$$-\quad \frac{\partial}{\partial y}\frac{\partial u}{\partial t}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+w\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}-f\frac{\partial v}{\partial y}-v\frac{\partial f}{\partial y}=-\cancel{\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}}+\frac{1}{\rho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\frac{df}{dt}=\cancel{\frac{\partial f}{\partial t}}+u\cancel{\frac{\partial f}{\partial x}}+v\frac{\partial f}{\partial y}+w\cancel{\frac{\partial f}{\partial z}}$$

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+\zeta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

Thus the vorticity equation,

$$\frac{d}{dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

Physical intuition?

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

Terms in vorticity equation

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

A **B** **C** **D**

A: Rate of change of absolute vorticity following the fluid motion

B: Effect of horizontal velocity divergence on vorticity

C: Transfer of vorticity between horizontal and vertical components
("twisting term" or "tilting term")

D: Effects of baroclinicity ("solenoidal term")

For pressure coordinates, solenoidal term disappears

$$\frac{d\zeta}{dt} = -\mathbf{V}\cdot\nabla(\zeta + f) - \omega\frac{\partial \zeta}{\partial p} - (\zeta + f)\nabla\cdot\mathbf{V} + \mathbf{k}\cdot\left(\frac{\partial \mathbf{V}}{dp}\times\nabla\omega\right)$$

In practice, solenoidal quite term is small in height coordinates too

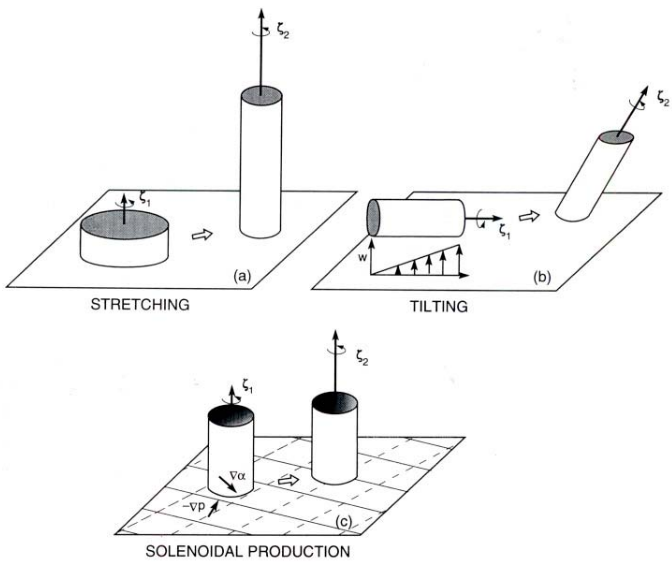


Figure 12.9 Forcing of the absolute vorticity $(f + \zeta)$ of a material element by (a) vertical stretching, which is compensated by horizontal convergence, (b) tilting, which exchanges horizontal and vertical vorticity, and (c) solenoidal production, which results from variation of density (shaded) across the pressure gradient force. The latter exerts a torque on the material element that is located inside a *solenoid* defined by two intersecting isobars (solid lines) and isochores (dashed lines).

Absolute vorticity (term A)

Expanding, the rate of change of absolute vorticity is written

$$\frac{d(\zeta+f)}{dt}=\frac{\partial(\zeta+f)}{\partial t}+u\frac{\partial(\zeta+f)}{\partial x}+v\frac{\partial(\zeta+f)}{\partial y}+w\frac{\partial(\zeta+f)}{\partial z}$$

local tendency of absolute vorticity	horizontal advection of absolute vorticity	vertical advection of absolute vorticity

$f = 2\Omega \sin\phi$. Being independent of x and z, we can write

$$\frac{d(\zeta+f)}{dt}=\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial(\zeta+f)}{\partial y}+w\frac{\partial\zeta}{\partial z}$$

Absolute vorticity change due to (3d) advection of relative vorticity, and meridional advection of planetary vorticity

Stretching/divergence (term B)

Effect of horizontal velocity divergence on vorticity

$$-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Divergence $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) > 0$

•Vorticity will decrease if absolute vorticity is positive
•Vorticity will increase if absolute vorticity is negative.

Convergence $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) < 0$

- Vorticity will increase if absolute vorticity is positive
- Vorticity will decrease if absolute vorticity is negative.

Consider vorticity of a region being squished into a smaller area.
This mechanism is quite important for large-scale midlatitude systems.

Tilting (term C)

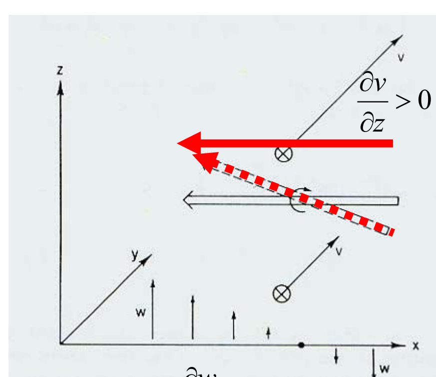
Transfer of vorticity between horizontal and vertical components

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

Vertical shear in of v wind is gives shear vorticity about an east-west axis. (consider "paddle" rotating around the solid vorticity vector – here ζ is zero)

East-west variations in the vertical velocity tilts the vector to be more vertical (dashed vorticity vector).

This new vector has a vertical component, and thus ζ is non zero.



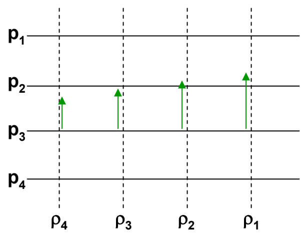
$$\frac{\partial w}{\partial x} < 0$$

$$\frac{\partial v}{\partial z}\frac{\partial w}{\partial x} < 0 \rightarrow \frac{d(\zeta + f)}{dt} > 0$$

Solenoidal term (term D)

Vorticity generation due to baroclinic structure
(density function of pressure and temperature)

$$\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$



$p_4 > p_3 > p_2 > p_1$

$\rho_4 > \rho_3 > \rho_2 > \rho_1$

For a uniform pressure gradient, horizontal variations density means there is a non-uniform acceleration due to the pressure gradient force.

$$\mathrm{PGF}_y(x) = -\frac{1}{\rho(x)}\frac{dp(y)}{dy}$$

Variations in acceleration produce vorticity $d\zeta/dt \sim d(\mathrm{PGF}_y)/dx$.

This can occur in a baroclinic atmosphere, but is absent in a barotropic atmosphere (or isentropic flow)

$$\frac{\partial p}{\partial y}<0;\ \frac{\partial \rho}{\partial x}<0 \rightarrow \frac{d(\zeta+f)}{dt}>0$$

Thought experiments

For dry adiabatic flow…

- The solenoid term disappears in pressure coordinates. Why?

$$\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)_p$$

Pressure variations along pressure surfaces… is zero

- The solidnoidal AND tilting term disappears in isentopic coordinates. Why?

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)_\theta$$

Adiabatic condition, means no vertical motion – so certainly no sheer in it!

Here the vertical velocity is now

$$w=\frac{d\theta}{dt}=\dot{\theta}$$

Recall vorticity advection

Advection of relative vorticity: $kU(k2+l2)$ A coskx cosly
Advection of planetary vorticity: $-bk$ A coskx cosly

- The advection of relative vorticity dominates for short waves
- Advection of planetary vorticity dominates for small waves
  (thus confirming what was seen in the project, small waves move slower when there is only advection)

- Given a westward mean flow, expect, short waves (individual lows, etc) tend to move eastward, while longer waves tend to move westward
  (in proactive long wave linked to topography and land se contrast, to tend not to move very much at all)
- *This can be all seen with the barotopic model*