

University of Colorado at Boulder

CU: Home | Search | A to Z | Map



Contact Us
 Search this Site
 Libraries Home

Research Resources | About the Libraries | Services **Chinook Catalog** Keyword

Search

Circulation Services

norcirc@colorado.edu
 303-492-7477 (ph)
 303-492-4560 (fx)
 Norlin, 1st Floor SE

- Home
- Hours
- Map
- Staff

FAQs

- Who may borrow books?
- How can I renew my books?
- Are books subject to recall?
- Where do I return books?
- Who resolves billing problems?

Library Login Information

- IdentiKey
- PIN Number
- SSN Policy

Services

- Borrowing Privileges
- Circulation Policy Manual
- Departmental Units
- e-Reserves (Information)
- Reserves via Chinook
- Team Technology Rooms

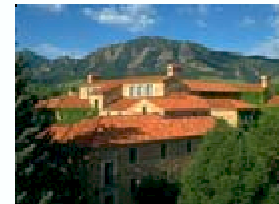
About the Libraries > Libraries & Departments >

e-Reserves: Cover Sheet

Overview | Faculty Guidelines | Copyright Guidelines | FAQ for Faculty | FAQ for Students | Troubleshooting Guide

Electronic Reserve Request

This cover sheet must accompany *each* item request.



TO: Norlin Library Reserves: 184 UCB

PROF.

David Noone

COURSE & # (e.g. Hist. 1243)

ATOC 5720 - Intro. to Atmospheric

SEMESTER

Fall

AUTHOR

R. R. Rogers and M. K. Yau

TITLE

A Short Course in Cloud Physics



UCB Libraries owns a copy of the original

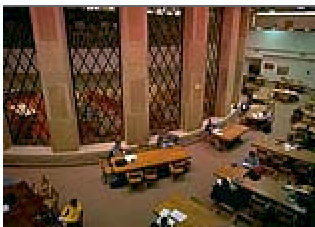


UCB Libraries does not own a copy of the original



I don't know if the UCB Libraries owns a copy of the original.

Please be sure to complete *all* lines above. Keep a copy of each file you send us. At the end of each semester, we delete all items from the reserve listings and from the reserve server.



University Libraries, 184 UCB, 1720 Pleasant Street, University of Colorado, Boulder, CO 80309-0184



Libraries Information 303-492-8705 | Reference & Instructional Services 303-492-7521
[Send comments to the WebMaster](#)
© 2004, The Regents of the University of Colorado

before embarking on the journey through theory. The central chapters, 6–10, are on the microphysics of clouds and precipitation. Here we have attempted, by focusing on fundamentals, to equip the student with what is needed to understand current journal articles or advanced books on these subjects. The later chapters move from microphysics to considerations of a larger scale. Chapter 11 is a brief introduction to meteorological radar, the instrument that has become essential in cloud physics research. Chapters 12 and 13, on precipitation processes and severe storms, show how radar observations are used to increase our understanding of cloud dynamics and microphysics. Chapter 14, another brief one, is on weather modification, a subject that has become somewhat detached from the core of cloud physics, but which has inspired so much activity in cloud physics that it should be included in even a short course. The most important recent advances in cloud dynamics have come from the computer simulation of clouds, the topic of the last chapter. Only through numerical modeling has it proven possible to understand the complex coupling between dynamics and microphysics, to make the detailed connections between theory and observations.

The book thus starts with the classical subject of thermodynamics and ends with numerical cloud modeling, an area of active current research in which new developments are rapidly emerging. We have tried throughout to be selective while covering the fundamentals, but ample references are included to enable the student to follow up details.

1 Thermodynamics of Dry Air

Atmospheric composition

Air is a mixture of several so-called permanent gases, a group of gases with variable concentrations, and different solid and liquid particles of variable concentrations. Nitrogen and oxygen account respectively for about 78% and 21% by volume of the atmosphere's permanent gases, with the remaining 1% consisting mainly of argon, but with trace amounts of neon, helium, and other gases. The composition of air is remarkably uniform, with the relative proportions of these permanent gases being essentially the same the world over and up to an altitude of 90 km.

The most abundant of the gases present in variable amounts are water vapor, carbon dioxide, and ozone. These gases strongly affect radiative transfer in the atmosphere. Water vapor is also of central importance in atmospheric thermodynamics.

The particles of solid and liquid material suspended in the air are called aerosols. Common examples are smoke, dust, and pollen. The water droplets and ice crystals of which clouds are composed are aerosols too, but they are usually classified with rain and other precipitation forms as *hydrometeors*, condensed forms of water in the atmosphere. Thermodynamics is concerned with the gases, but a select group of the aerosols called hygroscopic nuclei are crucial for the condensation of water in the atmosphere.

The approach in meteorology is to treat air as a mixture of two ideal gases: “dry air” and water vapor. This mixture is called moist air. The thermodynamic properties of moist air are determined by combining the separate thermodynamic behaviors of dry air and water vapor.

Equation of state for dry air

The equation of state for a perfect gas, or ideal gas law, expresses the relationship among pressure p , volume V , and temperature T of a gas in thermal equilibrium:

$$pV = CT, \quad (1.1)$$

with C a constant depending upon the particular gas.

The equation is reduced to standard form by employing Avogadro's Law, which states that at the same pressure and temperature, one mole of any gas occupies the same volume. Denoting this volume by v , we have

$$pv = C'T, \quad (1.2)$$

where C' is the same constant for all gases. It is called the *universal* gas constant, denoted by R^* , and equal to $8.314 \text{ J mole}^{-1} \text{ K}^{-1}$.

Since an arbitrary volume $V = nv$, with n the number of moles, it follows from (1.2) that

$$pV = nR^*T. \quad (1.3)$$

Dividing by the mass M of the gas gives

$$\frac{pV}{M} = \frac{n}{M} R^*T.$$

But $V/M = \alpha$, the specific volume, and $n/M = 1/m$ where m denotes the molecular weight of the gas. Consequently (1.3) reduces to

$$p\alpha = R'T, \quad (1.4)$$

where $R' = R^*/m$ is called the *individual* gas constant.

It is possible to calculate the effective molecular weight of dry air by suitably averaging the molecular weights of the nitrogen, oxygen, and trace gases of which it is composed. This turns out to be 28.96 g/mole . Accordingly, the individual gas constant for dry air is

$$R' = 287 \text{ J kg}^{-1} \text{ K}^{-1}.$$

Over the meteorological range of temperature and pressure, (1.4) describes the behavior of dry air with sufficient accuracy for most purposes.

For consistent SI units in (1.4), pressure is in Pa, specific volume in m^3/kg , and temperature in K. It is conventional in meteorology, however, to measure pressure in kilopascals ($1 \text{ kPa} = 10^3 \text{ Pa}$) or millibars ($1 \text{ mb} = 10^3 \text{ dynes/cm}^2 = 10^2 \text{ Pa} = 1 \text{ hPa} = 10^{-1} \text{ kPa}$). Temperature is often measured in degrees Celsius, related to degrees Kelvin by $T(\text{in K}) = T(\text{in } ^\circ\text{C}) + 273.15$.

The first law of thermodynamics

The first law is a statement of two empirical facts:

1. Heat is a form of energy.
2. Energy is conserved.

The first of these is called Joule's Law, and expresses the mechanical equivalent of heat as

$$1 \text{ cal} = 4.1868 \text{ J}. \quad (1.5)$$

The second of these empirical facts may be stated in algebraic form:

$$dQ = dU + dW. \quad (1.6)$$

Of the total amount of heat added to a gas, dQ , some may tend to increase the internal energy of the gas by amount dU , and the remainder will cause work to be done by the gas in the amount dW . It is generally more useful to express this relation for a unit mass of gas, for which (1.6) becomes

$$dq = du + dw. \quad (1.7)$$

We examine first the work term in (1.7). Consider a parcel of gas with volume V and surface area A , as illustrated in Fig. 1.1. The change in volume associated with a small incremental linear expansion dn is

$$dV = Adn.$$

But $p = F/A$, where F is the force exerted by the gas, so that

$$pdV = Fdn. \quad (1.8)$$

The work done by the gas in expanding is $dW = Fdn$. Consequently (1.8) may be written

$$dW = pdV.$$

The work done per unit mass of gas (specific work) is

$$dw = pda. \quad (1.9)$$

In general, the specific work done in a finite expansion from α_1 to α_2 is

$$\int dw = \int_{\alpha_1}^{\alpha_2} pda.$$

This integration may be visualized with the help of a thermodynamic diagram.

A thermodynamic diagram is a chart whose coordinates are variables of state. A given equilibrium thermodynamic state of a gas may be

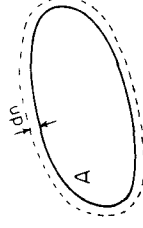


FIG. 1.1. Expanding parcel of gas.

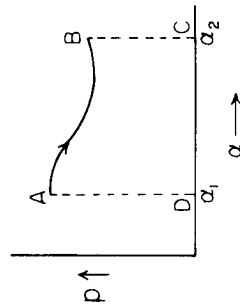


FIG. 1.2. Thermodynamic chart representing work done in expansion.

represented by a point on such a chart. As a gas goes through successive equilibrium states (for example in response to heating or to an external force), it traces out a path on a thermodynamic diagram.

The work done by a gas in expanding is readily illustrated on a chart with coordinates of pressure versus specific volume, as in Fig. 1.2. In the example shown the gas expands from initial state $A(p_1, \alpha_1)$ to final state $B(p_2, \alpha_2)$. The specific work done is represented by the area $ABCD$. There are actually any number of possible equilibrium paths from A to B , depending upon whether heat is added to or taken from the gas, and at what point during the process this heat transfer occurs. The work depends on the path of integration, which is another way of saying $dw = p d\alpha$ is not an exact differential.

Of special interest in thermodynamic theory are *cyclic* processes, in which the gas undergoes a continuous series of changes in state, but ends up with the same thermodynamic coordinates it had initially. One such cyclic process is pictured in Fig. 1.3 on a p, α -diagram. The gas starts at state A and proceeds to state B along the indicated curve. As before, the area under this curve gives the work done by the gas in expanding from α_1 to α_2 . Next the gas is compressed and made to return to state A along the lower curve. In this step of the process work is done *on* the gas. The net work done by the gas in the complete cyclic process is given by the hatched area. Note that if the process had taken place in the opposite sense, with arrows reversed, the hatched area would stand for net work

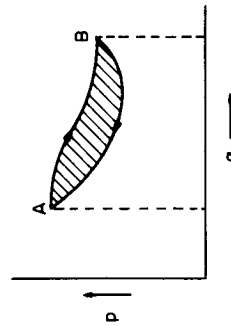


FIG. 1.3. Cyclic process.

done *on* the gas. The net work in a cyclic process is described mathematically by an integral over the closed path,

$$\oint dw = \oint p d\alpha.$$

For integrands that are exact differentials, any such cyclic integration yields zero, because the integral over an exact differential depends only on the limits of integration. This is not the case in general for dw , as already explained.

Next we consider the du term in (1.7). For an ideal gas, any increase in internal energy appears as an increase in temperature. The temperature change is proportional to the amount of heat added according to

$$dT = \frac{1}{c} dq, \quad (1.10)$$

where c is called the specific heat capacity and is measured, for example, in $\text{J kg}^{-1} \text{K}^{-1}$ or $\text{cal g}^{-1} \text{K}^{-1}$. For a gas, c is not constant but depends upon whether work is done while the heat is added. If no work is done, $d\alpha = 0$ from (1.9) and for the specific heat we write

$$c_v = \left(\frac{dq}{dT} \right)_\alpha, \quad (1.11)$$

known as the specific heat at constant volume.

Another case of interest is the addition of heat with pressure held constant, for which process the specific heat is given as

$$c_p = \left(\frac{dq}{dT} \right)_p, \quad (1.12)$$

and called the specific heat at constant pressure.

Evidently $c_p > c_v$, because in a constant pressure process some of the added heat will be used in the work term $p d\alpha$, while in the constant volume process all added heat goes toward increasing T . For dry air,

$$c_p = 1005 \text{ J kg}^{-1} \text{K}^{-1} = 0.240 \text{ cal g}^{-1} \text{K}^{-1}$$

$$c_v = 718 \text{ J kg}^{-1} \text{K}^{-1} = 0.171 \text{ cal g}^{-1} \text{K}^{-1}.$$

Of the total heat added, the amount that goes into the internal energy is

$$du = c_v dT \quad (1.13)$$

and the remainder goes into the work term. Thus the general expression for the conservation of energy is

$$dq = c_v dT + p d\alpha. \quad (1.14)$$

Special processes

By differentiating (1.4), we obtain

$$pda + \alpha dp = R'dT \quad (1.15)$$

as a differential equation relating changes of pressure, specific volume, and temperature under conditions of thermodynamic equilibrium. Combining (1.15) and (1.14),

$$dq = (c_v + R')dT - \alpha dp.$$

But

$$c_p = \left(\frac{dq}{dT} \right)_p = c_v + R',$$

so that

$$dq = c_p dT - \alpha dp, \quad (1.16)$$

which may be used instead of (1.14) as an expression of the first law.

Certain special processes are now defined using these equations.

(a) Isobaric process: $dp = 0$

$$dq = c_p dT = \left(\frac{c_p}{c_v} \right) c_v dT = \left(\frac{c_p}{c_v} \right) du. \quad (1.17)$$

(b) Isothermal process: $dT = 0$

$$dq = -\alpha dp = p d\alpha = d\omega. \quad (1.18)$$

(c) Isochoric process: $d\alpha = 0$

$$dq = c_v dT = du. \quad (1.19)$$

(d) Adiabatic process: $dq = 0$

$$c_p dT = \alpha dp \quad (1.20)$$

$$c_v dT = -p d\alpha. \quad (1.21)$$

or

The adiabatic process is of special significance because many of the temperature changes that take place in the atmosphere can be approximated as adiabatic. From (1.20) and the equation of state,

$$c_p dT = R'T \frac{dp}{p}, \quad (1.22)$$

which may be integrated to give

$$\left(\frac{T}{T_0} \right) = \left(\frac{p}{p_0} \right)^k, \quad (1.23)$$

where $k = R'/c_p = (c_p - c_v)/c_p = 0.286$.

The result (1.23) is called Poisson's equation for adiabatic processes. It is possible to derive expressions equivalent to (1.23) relating any two of the thermodynamic variables pressure, temperature, and specific volume.

A fourth thermodynamic variable, called the potential temperature, is defined on the basis of (1.23). It is denoted by θ and defined by

$$\left(\frac{T}{\theta} \right) = \left(\frac{p}{100 \text{ kPa}} \right)^k$$

$$\theta = T \left(\frac{100 \text{ kPa}}{p} \right)^k \quad (1.24)$$

or

and may be interpreted as the temperature that a parcel of air would have if, starting with temperature T at pressure p , it were subjected to an adiabatic compression or expansion to a final pressure of 100 kPa. The potential temperature is called a variable of state, because it is expressible in terms of the state variables p and T . In any adiabatic process, θ is a constant. We say that potential temperature is a *conservative* property in adiabatic processes.

Entropy

The second law of thermodynamics implies the existence of another variable of state, called entropy, which may be defined by the equation

$$d\phi = \frac{dq}{T}, \quad (1.25)$$

where $d\phi$ is the increase in (specific) entropy accompanying the addition of heat dq to a unit mass of gas at temperature T . It follows from (1.16) that

$$d\phi = \frac{1}{T} [c_p dT - \alpha dp] = c_p \frac{dT}{T} - R' \frac{dp}{p} = c_p \left[\frac{dT}{T} - k \frac{dp}{p} \right]$$

$$= c_p \frac{d\theta}{\theta}. \quad (1.26)$$

Integration gives

$$\phi = c_p \ln \theta + \text{const.}, \quad (1.27)$$

which associates entropy with potential temperature. It is evident from the defining equation (1.25) that adiabatic processes ($dq = 0$) are also isentropic processes.

Meteorological thermodynamic charts

(a) Stüve diagram

The Stüve diagram (or simply "adiabatic chart") is a thermodynamic diagram based on the adiabatic equation (1.24). This equation shows that, for a given value of θ , there is a linear relation between T and p^k . Consequently, adiabatic processes follow straight line paths on a thermodynamic diagram with coordinates of T versus p^k .

This kind of chart is convenient for depicting adiabatic processes in the atmosphere. A line along which $\theta = \text{const.}$ is called an *adiabat*. Figure 1.4 is a schematic diagram of a Stüve chart, illustrating the working coordinates of pressure and temperature, and also the appearance of isobars, adiabats, and isotherms.

(b) Emagram

So-called *true* thermodynamic diagrams are ones on which area is proportional to energy. Thus a p, α -diagram is a true thermodynamic diagram, because the area in any closed contour is proportional to the work done in a cyclic process defined by the contour.

In meteorology the state variables most frequently employed to describe the air are pressure and temperature. It is possible to construct a true thermodynamic diagram with coordinates of p and T on the basis of (1.9) and (1.15). We have

$$dw = pda = R'dT - \alpha dp$$

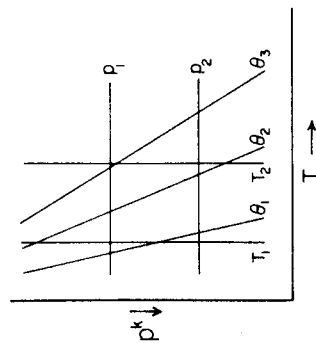


FIG. 1.4. Stüve diagram.

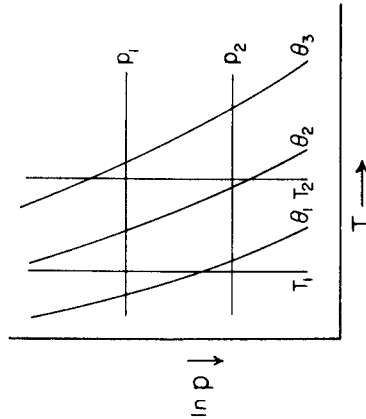


FIG. 1.5. Schematic emagram.

and, for a cyclic process,

$$\oint dw = \oint R'dT - \oint R'T \frac{dp}{p} \tag{1.28}$$

But $R'dT$ is an exact differential which integrates to zero, so that the work done reduces to

$$\oint dw = -R' \oint Td(\ln p) \tag{1.29}$$

This result indicates that a chart with coordinates of T versus $\ln p$ has the property of a true thermodynamic diagram. Such a chart is called an emagram, an abbreviation for energy-per-unit-mass diagram, and is illustrated schematically in Fig. 1.5.

(c) Tephigram

From the defining equation of entropy, it follows that the total heat added in a cyclic process is

$$\oint dq = \oint Td\phi = c_p \oint Td(\ln \theta) \tag{1.30}$$

Consequently a chart with coordinates of T versus ϕ , or equivalently T versus $\ln \theta$, has the required area-energy relation of a true thermodynamic diagram. Such a chart is called a tephigram, standing for T, ϕ -diagram, and is shown schematically in Fig. 1.6.

The tephigram is usually rotated so that the isobars end up more or less horizontal with pressure decreasing upwards on the chart. Figure 1.7 illustrates a chart with this orientation. It is based on the Canadian

meteorological service tephigram and will be used in the examples that follow.

Problems

1.1. The following table gives the approximate percentages by mass of the atmosphere's main permanent gases. Using these data, show that the effective molecular weight of dry air is 28.96 g/mole.

Gas	Mol. wt.	Mass, %
Nitrogen	28.016	75.57
Oxygen	32.000	23.15
Argon	39.944	1.28

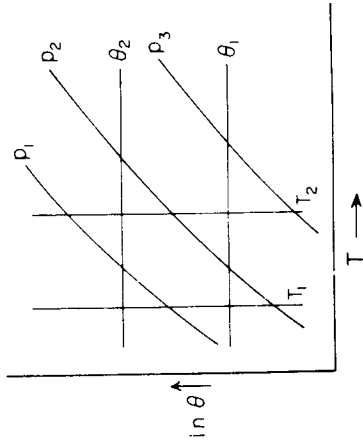


FIG. 1.6. Schematic tephigram.

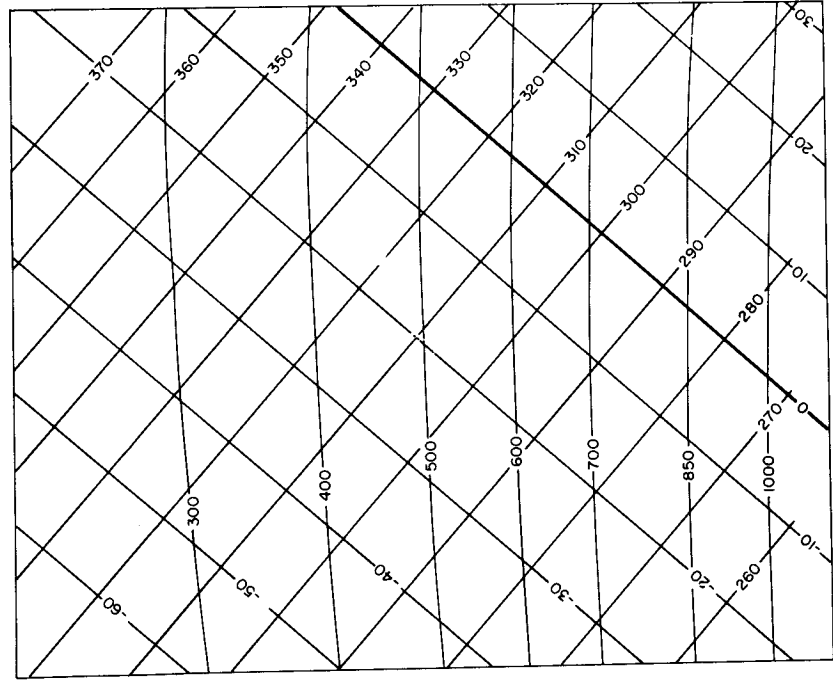


FIG. 1.7. Skeleton of a tephigram. Isobars are approximately horizontal, labeled in mb. Isotherms in deg C go upwards and to the right. Dry adiabats are normal to the isotherms and are labeled according to potential temperature (deg K).

1.2. A unit mass of dry air undergoes a Carnot cycle consisting of the following steps:

- (a) adiabatic compression from 60 kPa and 0°C to a temperature of 25°C;
- (b) isothermal expansion to a pressure of 70 kPa;
- (c) adiabatic expansion to a temperature of 0°C;
- (d) isothermal compression to the original pressure of 60 kPa.

Calculate the work done by the air in this process. Confirm your result using the tephigram or another meteorological thermodynamic chart.

1.3. A polytropic process for an ideal gas is one in which pressure and volume are related by $pV^n = \text{const.}$, where n is a constant. It is a generalization of the special processes considered earlier. Thus $n = 0$ defines an isobaric process, $n = c_p/c_v$ an adiabatic process, $n = 1$ an isothermal process, and $n = \infty$ an isochoric process. Suppose 1 kg of dry air at 280 K and 100 kPa undergoes a polytropic expansion in which the pressure falls to 70 kPa and the potential temperature increases by 10 K. Solve for

- (a) the value of n ;
- (b) the change in internal energy of the air;
- (c) the work done by the air;
- (d) the heat absorbed by the air.

1.4. The coordinates x and y of any thermodynamic chart are of the form $x = x(p, \alpha)$ and $y = y(p, \alpha)$. For example, an emagram has $x = p\alpha/R = T$ and $y = -R' \ln p$. A true thermodynamic diagram is one whose coordinates (x, y) may be transformed to (p, α) with a Jacobian of unity. That is, a given thermodynamic chart is a true thermodynamic diagram if

$$J \left(\frac{x, y}{p, \alpha} \right) = 1.$$

- (a) Prove that a Stüve diagram is not a true thermodynamic diagram.
- (b) How would you construct a true thermodynamic diagram with working coordinates of temperature and density? Indicate such a diagram schematically and sketch the approximate appearance of isobars and dry adiabats.

1.5. The specific enthalpy of a gas is defined by $h = u + p\alpha$.

(a) Prove that dh for an ideal gas is an exact differential.

(b) Calculate the change in enthalpy of a unit mass of dry air as it is compressed adiabatically from an initial pressure of 70 kPa and temperature of 10°C to a final pressure of 100 kPa.

1.6. A 200-gram sample of dry air is heated isobarically. Its entropy increases by 19.2 J/K and the work done by expansion is 1.61×10^3 J. Solve for the final temperature of the air.

temperature an equilibrium condition will eventually be reached when the two processes have the same rate. Then the temperature of the air and vapor equals that of the liquid and there is no net transfer of molecules from one phase to the other. The space above the liquid is then said to be *saturated* with water vapor. The partial pressure due to the water vapor in this condition is called the saturation vapor pressure. It is found that the saturation vapor pressure depends only on temperature, and this functional dependence is described by an important differential equation that we will now derive.

Heat is required to change phase from liquid to vapor because the kinetic energy of the vapor molecules exceeds that of liquid molecules at the same temperature. We denote by L the heat required to convert a unit mass of liquid to vapor, with pressure and temperature held constant. This is the latent heat of vaporization. For this transition, from phase 1 (liquid) to phase 2 (vapor),

$$L = \int_{q_1}^{q_2} dq = \int_{u_1}^{u_2} du + \int_{\alpha_1}^{\alpha_2} p d\alpha = u_2 - u_1 + e_s(\alpha_2 - \alpha_1), \quad (2.3)$$

where e_s denotes the saturation vapor pressure, which is constant throughout the process. Because the temperature is also constant, we may write

$$L = T \int_{q_1}^{q_2} \frac{dq}{T} = T(\phi_2 - \phi_1). \quad (2.4)$$

Equating results, we find that

$$u_1 + e_s \alpha_1 - T\phi_1 = u_2 + e_s \alpha_2 - T\phi_2, \quad (2.5)$$

which shows that this particular combination of thermodynamic variables remains constant in an isothermal, isobaric change of phase. This combination is called the Gibbs function of the system and is denoted by G . Thus, for phase 1,

$$G_1 = u_1 + e_s \alpha_1 - T\phi_1 \quad (2.6)$$

and (2.5) may be written as $G_1 = G_2$.

Though it is constant in the phase transition, the Gibbs function varies with temperature and pressure, and its dependence on these variables may be determined by differentiation:

$$dG = du + e_s d\alpha + \alpha de_s - T d\phi - \phi dT. \quad (2.7)$$

But $du + e_s d\alpha = dq = T d\phi$, and (2.7) reduces to

$$dG = \alpha de_s - \phi dT. \quad (2.8)$$

Because G is the same for both phases, $dG_1 = dG_2$ and (2.8) implies

2 Water Vapor and its Thermodynamic Effects

Equation of state for water vapor

Unlike other atmospheric constituents, water appears in all three phases, solid, liquid, and vapor. In the vapor phase water in the atmosphere behaves as an ideal gas to a good approximation. Its equation of state is

$$e = \rho_v R_v T, \quad (2.1)$$

where e = vapor pressure, ρ_v = vapor density, and R_v = individual gas constant for water vapor ($461.5 \text{ J kg}^{-1} \text{ K}^{-1}$). This equation sometimes appears in the form

$$e = \rho_v \frac{R'}{\epsilon} T, \quad (2.2)$$

where $\epsilon = R'/R_v = m_v/m = 0.622$.

Clausius-Clapeyron equation

Consider a closed and thermally insulated container partly filled with water as shown in Fig. 2.1. Molecules from the surface layer of water are in agitation and some break away as vapor molecules. On the other hand some of the vapor molecules collide with the surface and stick. Condensation and evaporation thus take place simultaneously. For a given

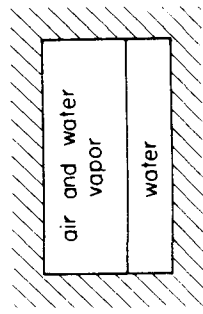


FIG. 2.1. Vapor in equilibrium with liquid surface.

$$\frac{de_s}{dT} = \frac{\phi_2 - \phi_1}{\alpha_2 - \alpha_1} = \frac{L}{T(\alpha_2 - \alpha_1)} \quad (2.9)$$

This result expresses the change of saturation vapor pressure with temperature and is known as the Clausius-Clapeyron equation. Under ordinary atmospheric conditions $\alpha_2 \gg \alpha_1$ and water vapor behaves as an ideal gas. Then (2.9) reduces to

$$\frac{de_s}{dT} = \frac{L}{T\alpha_2} = \frac{Le_s}{R_v T^2} \quad (2.10)$$

At temperatures colder than 0°C , (2.10) describes the saturation vapor pressure of supercooled liquid water. Ice may also exist and be in equilibrium with the vapor at subfreezing temperatures. The change with temperature of the saturation vapor pressure of ice is given by the Clausius-Clapeyron equation, except with L in (2.10) replaced by L_s , the latent heat of sublimation. At temperatures warmer than 0°C , only liquid water can be in equilibrium with the vapor.

As a first approximation, the Clausius-Clapeyron equation can be integrated by regarding the latent heat as constant. The result is

$$\ln \frac{e_s(T)}{e_{s0}} = \frac{L}{R_v} \left(\frac{1}{T_0} - \frac{1}{T} \right), \quad (2.11)$$

where e_{s0} is the value of saturation vapor pressure at temperature T_0 , a constant of integration that must be determined by experiment. It is found that $e_{s0} = 611 \text{ Pa}$ at $T_0 = 0^\circ\text{C}$. The latent heat of vaporization near 0°C is approximately $2.50 \times 10^6 \text{ J/kg}$. Substituting these values in (2.11) gives, as an approximation for the saturation vapor pressure over water,

$$e_s(T) = Ae^{-B/T}, \quad (2.12)$$

where $A = 2.53 \times 10^8 \text{ kPa}$ and $B = 5.42 \times 10^3 \text{ K}$.

The latent heat of vaporization depends weakly on temperature, changing by about 6% over the temperature range from -30°C to $+30^\circ\text{C}$. This dependence may be inferred from (2.3). We start by noting that in this equation $\alpha_2 \gg \alpha_1$, and $e_s \alpha_2 = R_v T$. Then, differentiating with respect to T , we find

$$\frac{dL}{dT} = c_{vw} - c + R_v = c_{pv} - c \quad (2.13)$$

where $c_{vw} = du_2/dT$ is the specific heat capacity of water vapor at constant volume; $c = du_1/dT$ is the specific heat capacity of liquid water; and $c_{pv} = R_v + c_{vw}$ is the specific heat of water vapor at constant pressure. Regarding the specific heats as constants, we may integrate (2.13) and write

$$L(T) = L_0 - (c - c_{pv})(T - T_0), \quad (2.14)$$

where $L_0 = L(T_0)$ is a constant of integration. A more accurate approximation than (2.12) for $e_s(T)$ can then be obtained by substituting (2.14) in the Clausius-Clapeyron equation and integrating (see problem 2.5).

The specific heat capacities also depend on temperature and pressure, although very weakly. At the saturation vapor pressure, for example, the specific heat at constant pressure c_{pv} increases with increasing temperature and is about 2% larger at $+30^\circ\text{C}$ than at -30°C . For many purposes this variation can be neglected, and the value of c_{pv} can be taken as $1870 \text{ J kg}^{-1} \text{ K}^{-1}$. A good approximation for the specific heat at constant volume is $1410 \text{ J kg}^{-1} \text{ K}^{-1}$. For liquid water, the specific heat is within 1% of $1 \text{ cal g}^{-1} \text{ K}^{-1} = 4187 \text{ J kg}^{-1} \text{ K}^{-1}$ at temperatures warmer than 0°C , but as the temperature falls below 0°C this quantity gradually increases to a value approximately 8% larger at -30°C .

A first approximation for the saturation vapor pressure e_i of ice is (2.12), with $A = 3.41 \times 10^9 \text{ kPa}$ and $B = 6.13 \times 10^3 \text{ K}$. This approximation follows from replacing L in (2.11) by L_s , the latent heat of sublimation, and using $2.83 \times 10^6 \text{ J kg}^{-1}$ as the value of L_s and 611 Pa as the value of e_i at 0°C . (Experiments show that e_s and e_i are equal to within three significant figures at 0°C .)

Comparing (2.11) for water and ice shows that, at a given subfreezing temperature,

$$\frac{e_s(T)}{e_i(T)} = \exp \left\{ \frac{L_f}{R_v T_0} \left(\frac{T_0}{T} - 1 \right) \right\} \quad (2.15)$$

where $L_f = L_s - L$ is the latent heat of fusion of water. Numerically, in the vicinity of 0°C , a good approximation to this equation is

$$\frac{e_s(T)}{e_i(T)} \approx \left(\frac{273}{T} \right)^{2.66}, \quad (2.16)$$

where T is in K. These relations indicate that the saturation vapor pressure of water exceeds that of ice for all temperatures below 273 K and that the ratio e_s/e_i steadily increases as the temperature decreases. Any atmosphere saturated with respect to water is supersaturated relative to ice; the degree of supersaturation increases with the supercooling.

Although the relationships given thus far are adequate for many purposes, accurate and precise values of the saturation vapor pressure are needed for some applications. Table 2.1 lists the accepted standard values of e_s and e_i over the range from -40°C to $+40^\circ\text{C}$. Also included are values of L and L_s . The vapor pressures were obtained by integrating the Clausius-Clapeyron equation, taking into account the most complete information on the dependence on temperature of the latent heats and

TABLE 2.1. Saturation Vapor Pressures Over Water and Ice, and Latent Heats of Condensation and Sublimation

$T(^{\circ}\text{C})$	$e_s(\text{Pa})$	$e_i(\text{Pa})$	$L(\text{J/g})$	$L_s(\text{J/g})$
-40	19.05	12.85	2603	2839
-35	31.54	22.36		
-30	51.06	38.02	2575	2839
-25	80.90	63.30		
-20	125.63	103.28	2549	2838
-15	191.44	165.32		
-10	286.57	259.92	2525	2837
-5	421.84	401.78		
0	611.21	611.15	2501	2834
5	872.47	2489	2477	
10	1227.94		2466	
15	1705.32		2453	
20	2338.54		2442	
25	3168.74		2430	
30	4245.20		2418	
35	5626.45		2406	
40	7381.27			

specific heats, and fitting the curves to the points where definitive experimental values are known. The values of e_s for $T \geq 0^{\circ}\text{C}$ are taken from Wexler (1976); those for e_i are from Wexler (1977). There is some uncertainty about the values of e_s for $T < 0^{\circ}\text{C}$ owing to a lack of experimental data. The entries here were obtained by extrapolating Wexler's formula to temperatures below 0°C . The values of latent heat were taken from the 6th edition (R. J. List, 1951) of the Smithsonian Meteorological Tables. Bolton (1980) has shown that the tabulated data on $e_s(T)$ for water can be fitted to within 0.1% over the temperature range $-30^{\circ}\text{C} \leq T \leq 35^{\circ}\text{C}$ by the empirical formula

$$e_s(T) = 6.112 \exp\left(\frac{17.67T}{T + 243.5}\right), \quad (2.17)$$

where e_s is in mb and T is in degrees C.

Moist air: its vapor content

Atmospheric air is a mixture of dry air and water vapor. There are different ways to describe the vapor content, depending on the application.

- (a) Vapor pressure e , the partial pressure of the water vapor.
- (b) Vapor density ρ_v , also called absolute humidity, defined by (2.1).
- (c) Mixing ratio w , defined as the mass of water vapor per unit mass of dry air.

$$w = M_v/M_d = \rho_v/\rho_d.$$

From the equation of state, $\rho_v = e/R_v T$ and $\rho_d = (p - e)/R' T$ so that

$$w = \epsilon \frac{e}{p - e} \approx \epsilon \frac{e}{p}. \quad (2.18)$$

(d) Specific humidity q , the mass of water vapor per unit mass of moist air.

$$q = \frac{\rho_v}{\rho} = \frac{\rho_v}{\rho_d + \rho_v} = \epsilon \frac{e}{p - (1 - \epsilon)e} \approx \epsilon \frac{e}{p}. \quad (2.19)$$

The saturation mixing ratio and saturation specific humidity, denoted by w_s and q_s , are defined by (2.18) and (2.19) by formally replacing e by e_s . Because $e_s = e_s(T)$, w_s and q_s are functions of temperature and pressure only, and do not depend on the vapor content of the air. All meteorological thermodynamic charts contain "vapor lines", which are usually isopleths of w_s .

(e) Relative humidity f , the ratio of the mixing ratio to its saturation value at the same temperature and pressure.

$$f = \frac{w}{w_s(p, T)} \approx \frac{e}{e_s}. \quad (2.20)$$

The relative humidity is usually expressed in per cent.

(f) Virtual temperature T_v , the temperature of dry air having the same density as a sample of moist air at the same pressure.

For a sample of air of volume V , having total pressure p and vapor pressure e ,

$$\begin{aligned} p &= p_d + e = \rho_d \frac{R^*}{m_d} T + \rho_v \frac{R^*}{m_v} T \\ &= \frac{R^* T}{V} \left[\frac{M_d}{m_d} + \frac{M_v}{m_v} \right] \\ &= \rho R^* T \left[\frac{M_d}{m_d} + \frac{M_v}{m_v} \right] \frac{1}{M_d + M_v} \\ &= \rho R' T \left[\frac{1 + w/\epsilon}{1 + w} \right]. \end{aligned}$$

This result indicates that the equation of state for dry air may be applied to moist air if we include the correction factor in brackets. The virtual temperature is introduced to include this correction factor.

$$T_v = T \left[\frac{1 + w/\epsilon}{1 + w} \right] \approx T[1 + 0.6w]. \quad (2.21)$$

Thermodynamics of unsaturated moist air

(a) Gas constant

The equation of state for dry air can be applied to moist air if we replace T by T_v . Thus

$$p\alpha = R'T_v \tag{2.22}$$

is a general equation of state applicable to dry or moist air. Frequently the difference between actual and virtual temperature is small and may be neglected.

Alternatively the equation of state for moist air can be written

$$p\alpha = R_m T, \tag{2.23}$$

where R_m is the individual gas constant for moist air, which must depend on the mixing ratio according to

$$R_m = R'[1 + 0.6w]. \tag{2.24}$$

Obviously (2.22) and (2.23) are equivalent; it only depends on where you choose to apply the correction factor.

(b) Specific heat

To determine c_{vm} , the specific heat at constant volume for moist air, consider the addition of heat to a sample of air consisting of one kilogram of dry air and w kilograms of water vapor.

$$(1 + w)dq = c_v dT + wc_v dT,$$

where c_v is the specific heat of dry air and c_{vw} is the specific heat of the vapor. This shows that

$$c_{vm} = \left(\frac{dq}{dT}\right) = c_v \left[\frac{1 + wr}{1 + w}\right],$$

where

$$r = c_{vw}/c_v = 1410/718 = 1.96.$$

Thus

$$c_{vm} \approx c_v[1 + w]. \tag{2.25}$$

The same procedure may be employed to show that the specific heat at constant pressure for moist air is

$$c_{pm} \approx c_p[1 + 0.9w]. \tag{2.26}$$

Combining (2.24) and (2.26), the exponent in the Poisson equation

(1.23) for adiabatic processes in moist air is found to be

$$\frac{R_m}{c_{pm}} \approx k[1 - 0.2w]. \tag{2.27}$$

Because w is of the order 10^{-2} or less, the correction factors in (2.24)–(2.27) may often be neglected.

Ways of reaching saturation

A sample of moist air may undergo several processes that lead to saturation. Some of these processes are of theoretical importance, and introduce certain new temperatures that reflect the moisture content of the air.

(a) Dew point temperature T_d , defined as the temperature to which moist air must be cooled, with p and w held constant, for it to reach saturation with respect to water. (The frost point temperature is defined similarly, except for saturation relative to ice.) Clearly the saturation mixing ratio at the dew point equals the mixing ratio of the moist air: $w = w_s(p, T_d)$. An analytical approximation for the dew point follows from (2.12):

$$T_d = T_d(w, p) = B/\ln(A\epsilon/wp). \tag{2.28}$$

Graphical determination of the dew point is illustrated in Fig. 2.2.

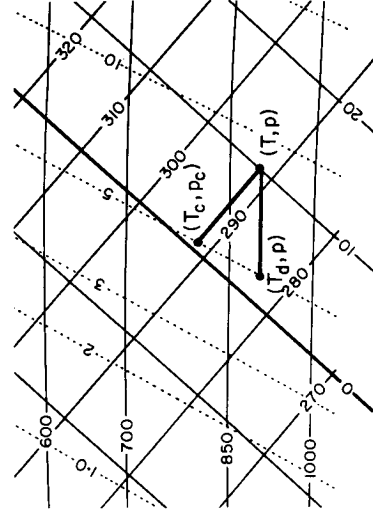


FIG. 2.2. Temperature, dew point, and isentropic condensation temperature, indicated on a tephigram. In the example shown the sample of air at 10°C, 900 mb, is assumed to have a mixing ratio of 5 g/kg. Its dew point, found from the intersection of the 900 mb isobar and the 5 g/kg vapor line, is 2.2°C. Its isentropic condensation point, found from the intersection of the adiabat through (T, p) with the 5 g/kg vapor line, is at 0.7°C and approximately 800 mb.

(b) Wet-bulb temperature T_w , defined as the temperature to which air may be cooled by evaporating water into it at constant pressure, until saturation is reached. (Note that w is not held constant, so that $T_d \neq T_w$ in general.)

Consider a sample of moist air consisting of one kilogram of dry air and w kilograms of water vapor. The first law of thermodynamics for this sample in an isobaric process is (from 2.26)

$$dq = c_p dT[1 + 0.9w].$$

Associated with the evaporation of mass dw of water is a heat loss given by

$$(1 + w)dq = -Ldw.$$

Consequently

$$c_p dT = -Ldw \left(\frac{1}{1+w} \right) \left(\frac{1}{1+0.9w} \right) \approx -Ldw[1 - 1.9w].$$

It is often satisfactory to neglect the correction factor, so that

$$c_p dT = -Ldw, \tag{2.29}$$

which approximately describes the wet-bulb process. Neglecting the weak dependence of L on temperature, this equation may be integrated to yield

$$\frac{T - T_w}{w_s(p, T_w) - w} = \frac{L}{c_p}. \tag{2.30}$$

This approximation, combined with the defining equation for $w_s(p, T)$ and the Clausius-Clapeyron equation for $e_s(T)$, may be used to solve for any one of the quantities T , T_w , or w , given the other two. For example, if we use the approximation (2.12) for $e_s(T)$, then T_w is given in terms of T and w by

$$T_w = T - \frac{L}{c_p} \left(\frac{\varepsilon}{p} A e^{-B/T_w} - w \right), \tag{2.31}$$

which may be solved by iteration.

(c) Equivalent temperature T_e , defined as the temperature a sample of moist air would attain if all the moisture were condensed out at constant pressure. An expression for T_e follows from (2.30) if we set $w_s = 0$ (the final mixing ratio) and $T_w = T_e$. Thus

$$T_e = T + \frac{Lw}{c_p}. \tag{2.32}$$

(d) Isentropic condensation temperature T_c , defined as the tempera-

ture at which saturation is reached when moist air is cooled adiabatically with w held constant. This temperature is most readily understood with the help of a thermodynamic chart (Fig. 2.2).

The air initially has coordinates (T, p) with mixing ratio w . It is cooled adiabatically until its adiabat intersects the vapor line defined by $w_s = w$. The pressure at this intersection is called the isentropic condensation pressure, and the temperature is T_c . An analytical approximation for T_c , which must be solved by iteration, is

$$T_c = B/\ln \left[\frac{A\varepsilon}{wp_0} \left(\frac{T_0}{T_c} \right)^{1/k} \right]. \tag{2.33}$$

This equation is obtained by noting that $T_c = T_d(w, p_c)$ and by substituting (2.28) for T_d into the adiabatic equation (1.23), written in the form

$$\frac{T_c}{T_0} = \left(\frac{p_c}{p_0} \right)^k.$$

Actually, it is not obvious that condensation should occur when expansion continues beyond the saturation point. Experience shows that this does happen in the atmosphere, so that we speak interchangeably of the condensation point and the saturation point.

Pseudoadiabatic process*

If expansion continues after the isentropic condensation point is reached, condensation occurs and the released latent heat will tend to warm the air. As a result, the temperature will decrease with falling pressure at a slower rate after condensation than before. To calculate the dependence of T on p in this process, assumptions must be made about the condensed water. Does it stay with the air in the form of cloud droplets or does it precipitate out? At subfreezing temperatures is the condensate water or ice? The different alternatives are compared in standard texts on dynamic meteorology. It turns out that the final result—the dependence of T on p —is not significantly affected by the choice of assumptions. The simplest case is the so-called pseudoadiabatic process in which the condensate is assumed to be water which immediately precipitates. This is the simplest case because the heat content of the condensed material need not be considered when calculating tem-

* In a brief history of early developments in the theory of the saturated adiabatic process, McDonald (1963a) explained that Lord Kelvin in 1862 gave the first correct description of the process; that Heinrich Hertz constructed an adiabatic diagram in 1884 that was the prototype of all subsequent meteorological thermodynamic charts; and that Wilhelm von Bezold in 1888 formulated the theory and equations for the pseudoadiabatic process.

perature changes of the air. Also, the question at what temperature sublimation becomes important is avoided.

Consider a sample of saturated air consisting of one kilogram of dry air and w_s kilograms of water vapor. Let its pressure change by amount dp . The temperature will change by dT and a corresponding change dw_s in vapor content will occur. The equation relating these incremental changes is, to good approximation,

$$\frac{dT}{T} = k \frac{dp}{p} - \frac{L}{Tc_p} dw_s, \tag{2.34}$$

a mathematical description of the pseudoadiabatic process. This formula is the basis of "pseudoadiabats" on a thermodynamic chart.

It is straightforward to employ an analytical approximation for $e_s(T)$, such as (2.12), and to express w_s in terms of differentials of temperature and pressure. When this expression is substituted in (2.34), the terms in dT and dp can be collected to solve for dT/dp in a pseudoadiabatic process. This relationship can then be integrated numerically to give temperature as a function of pressure along a pseudoadiabat.

Figure 2.3 illustrates the pseudoadiabatic expansion process. In an adiabatic expansion, the temperature decreases along a dry adiabat until the isentropic condensation point P is reached. Continued expansion is accompanied by the release of latent heat and the temperature follows along a pseudoadiabat from P onwards.

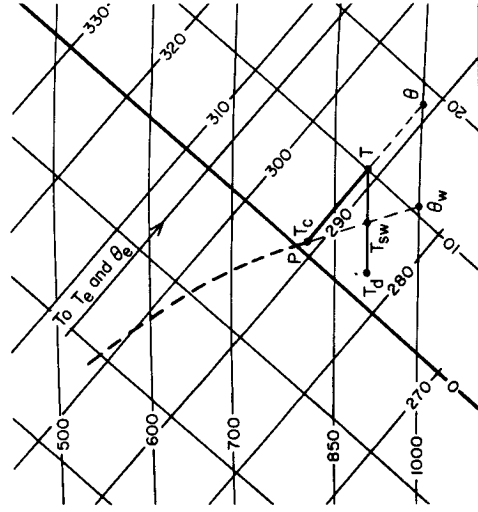


FIG. 2.3. Continued expansion of the air sample of Fig. 2.2 beyond point P , the isentropic condensation point. The dashed line is the pseudoadiabat through P . This diagram indicates the graphical determination of some of the important theoretical temperatures that characterize an air sample.

Some additional special temperatures that may be defined by referring to this illustration are:

- (a) Adiabatic wet-bulb temperature T_{sw} , obtained by following the pseudoadiabat from P down to the original pressure. The result is within 0.5°C of the wet-bulb temperature defined by (2.30).
- (b) Wet-bulb potential temperature θ_w , defined by the intersection of the pseudoadiabat through P with the isobar $p = 1000$ mb.
- (c) Equivalent temperature T_e (adiabatic definition), obtained by following up the pseudoadiabat from P to very low pressure, thus condensing out all the water vapor, and then returning to the original pressure along a dry adiabat. This temperature follows from integrating (2.34) from initial temperature T to final temperature T_e , and may be shown to be approximately

$$T_e = T \exp \left[\frac{Lw_s}{c_p T_c} \right]. \tag{2.35}$$

- (d) Equivalent potential temperature θ_e , defined as the temperature a parcel of air would have if taken from its equivalent temperature to a pressure of 1000 mb in a dry adiabatic process. A semi-empirical formula for θ_e , accurate to within 0.5 K, is

$$\theta_e = \theta \exp (2675w/T_c). \tag{2.36}$$

There is a one-to-one relationship between θ_e and θ_w . Both are determined by the process pseudoadiabat that characterizes the air sample; both are conservative in dry adiabatic or pseudoadiabatic processes. Highly accurate empirical formulas for these quantities are given by Bolton (1980).

The disposition of T , T_c , T_d , and T_{sw} about the perimeter of a triangle is called Normand's rule, after the British meteorologist Sir Charles Normand. It provides a helpful method of determining some of the thermodynamic properties of an air sample graphically.

Figure 2.4 is a tephigram including pseudoadiabats and vapor lines. It may be used for approximate graphical calculations.

Adiabatic liquid water content

In the pseudoadiabatic process the condensed phase is disregarded in calculating the changes in temperature of the air. The amount of this condensed material may be determined as follows. Suppose the air at the isentropic condensation point consists of 1 kg of dry air and w_s kg of water vapor. As this air expands pseudoadiabatically, the saturation mixing ratio decreases by the amount dw_s . If the air is to remain saturated, the

same amount of water must be condensed. Denoting this amount by $d\chi$, we have

$$d\chi = -dw_s \tag{2.37}$$

as the relation between the saturation mixing ratio and the amount of condensed material. We call $d\chi$ the increase in adiabatic liquid water mixing ratio. χ increases as expansion continues, and is given at any time by $\chi = \Delta w_s$, where Δw_s is the decrease in w_s from its value at the condensation point. Note that the mass of liquid water per unit mass of moist air is $\chi/(1 + w_s)$. The liquid water density, the mass per unit volume of air, equals $\rho\chi$, where ρ is the air density. All of these quantities are measures of the adiabatic liquid water content of air produced by pseudoadiabatic expansion beyond the condensation point.

Reversible saturated adiabatic process

In this process, the condensed water remains with the air and its heat capacity is taken into account. From (2.37) it follows that the total water mixing ratio Q , defined by

$$Q = w_s + \chi, \tag{2.38}$$

is conserved in a saturated air parcel. Moreover, on the assumption that the parcel is a closed thermodynamic system, the changes in temperature will be reversible, adiabatic, and isentropic.

The specific entropy of the cloudy air is given by

$$\phi = \phi_d + w_s\phi_v + \chi\phi_w, \tag{2.39}$$

where ϕ_d , ϕ_v , and ϕ_w are respectively the specific entropies of the dry air, the vapor, and the liquid. From (2.4), we note that $\phi_v = \phi_w + (L/T)$, so that (2.39) may be written

$$\phi = \phi_d + \phi_w Q + \frac{L}{T} w_s. \tag{2.40}$$

Therefore, in an isentropic process,

$$d\phi = 0 = d\phi_d + d(\phi_w Q) + d(Lw_s/T). \tag{2.41}$$

But

$$d\phi_d = c_p d(\ln T) - R' d(\ln p_d)$$

and

$$d\phi_w = c_w d(\ln T),$$

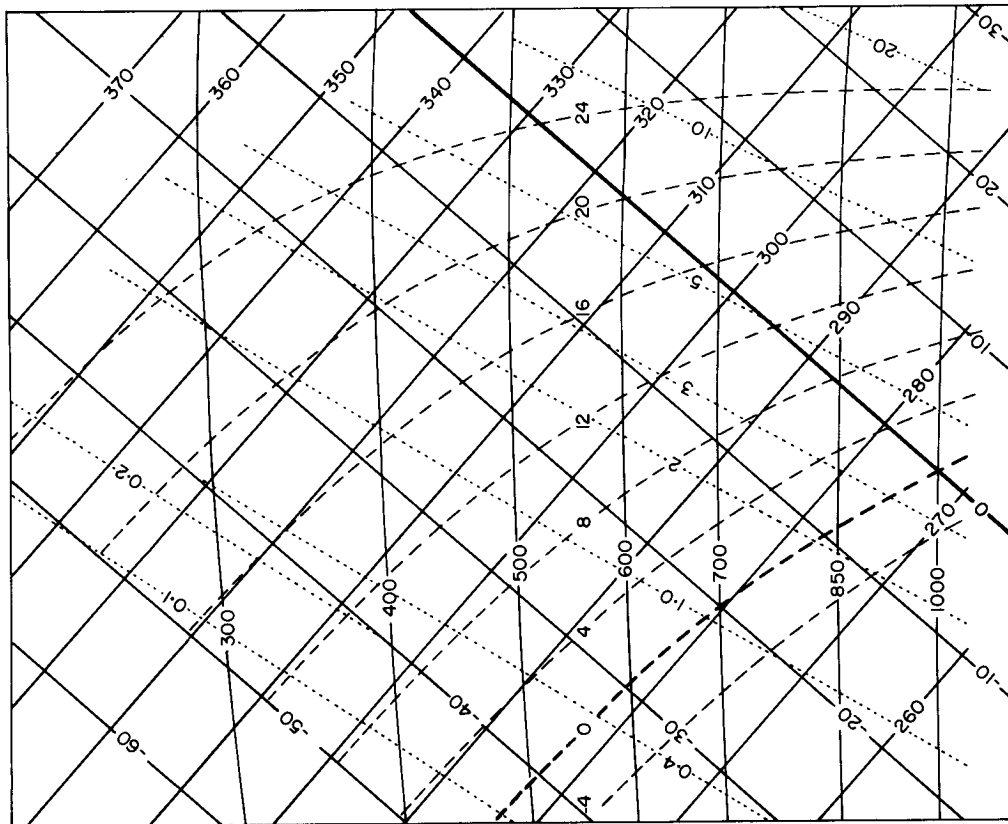


FIG. 2.4. Tephigram with pseudoadiabats and vapor lines. The pseudoadiabats are lines of constant θ_w and are labeled in degrees C at a position on the chart between 500 and 600 mb. The vapor lines are lines of constant w_s and are labeled in g/kg.

where c_w here stands for the specific heat of liquid water, so that

$$(c_p + Q_w)d(\ln T) - R'd(\ln p_d) + d(Lw_s/T) = 0. \quad (2.42)$$

The main difference between (2.42) and (2.34) is that the latent heat absorbed by the water substance is accounted for in a reversible saturated adiabatic process. For most applications, the two equations give very similar results for the change of temperature with pressure.

Integrating (2.42) gives the result

$$\frac{T}{p_d^{R'/(c_p + c_w Q)}} \exp \left[\frac{w_s L}{T(c_p + c_w Q)} \right] = \text{constant}.$$

By analogy with the definition of potential temperature, we can now define the wet equivalent potential temperature θ_q as

$$\theta_q(\text{sat.}) = T \left(\frac{100 \text{ kPa}}{p_d} \right)^{R'/(c_p + c_w Q)} \exp \left[\frac{w_s L}{T(c_p + c_w Q)} \right]. \quad (2.43)$$

θ_q is a conservative quantity in a reversible saturated adiabatic process.

If the air parcel is not saturated, it can be shown that

$$\theta_q(\text{unsat.}) = \kappa T \left(\frac{100 \text{ kPa}}{p_d} \right)^{(R' + R_s Q)/(c_p + c_w Q)} \quad (2.44)$$

is conserved along a dry adiabat. The constant κ can be evaluated by requiring that $\theta_q(\text{sat.}) = \theta_q(\text{unsat.})$ at the pressure level where condensation occurs.

Problems

- 2.1. Prove that pure water vapor can reach saturation in the following processes:
 - (a) isothermal compression;
 - (b) adiabatic expansion.
- 2.2. A sample of moist air has a temperature of -5°C , a pressure of 80 kPa, and a relative humidity of 65%. Solve for the following properties of the sample by calculations, using approximations where convenient. Confirm the answers with the tephigram wherever possible:
 - (a) potential temperature;
 - (b) mixing ratio;
 - (c) dew point;
 - (d) isentropic condensation temperature;
 - (e) wet-bulb temperature;
 - (f) wet-bulb potential temperature;
 - (g) equivalent temperature;
 - (h) virtual temperature;
 - (i) density.
- 2.3. Household humidifiers work by evaporating water into the air of a confined space and raising its relative humidity. A large room with a volume of 100 m^3 contains air at 23°C with a relative humidity of 15%. Compute the amount of water that must be

evaporated to raise the relative humidity to 65%. Assume an isobaric process at 100 kPa in which the heat required to evaporate the water is supplied by the air. Confirm your answer with the tephigram.

2.4. A pseudoadiabatic process is not a polytropic process but it may be approximated as such over a limited range of pressure and temperature. Show that in a pseudo-adiabatic process at 270 K and 80 kPa, pressure and specific volume are related by $p\alpha^n = \text{const.}$, where n has the value 1.23. Show further that n deviates only 5% from this value as the temperature changes by $\pm 10 \text{ K}$ about 270 K and the pressure changes by $\pm 10 \text{ kPa}$ about 80 kPa.

2.5. Using (2.14) for the dependence of the latent heat of vaporization on temperature, integrate (2.10) obtaining an expression for $e_s(T)$. (This result is sometimes called the Magnus equation or Kiefer's formula.) Over the range $-30^\circ\text{C} \leq T \leq 30^\circ\text{C}$, compare $e_s(T)$ from this equation with the simpler approximation (2.12), with Bolton's empirical formula (2.17), and with the data in Table 2.1.

over the pressure interval from p to p_0 , given by

$$\bar{T}_v = \frac{\int_{\ln p}^{\ln p_0} T_v d(\ln p)}{\ln p - \ln p_0}. \tag{3.4}$$

Parcel Buoyancy and Atmospheric Stability

Hydrostatic equilibrium

Air is said to be in hydrostatic equilibrium when it experiences no net force in the vertical direction. Then the vertical pressure gradient force on the air exactly balances the force of gravity (Fig. 3.1), and

$$\frac{\partial p}{\partial z} = -\rho g, \tag{3.1}$$

the well-known hydrostatic equation. Integration of (3.1) shows that the hydrostatic pressure at any level in the atmosphere is equal to the weight of an air column with unit cross-sectional area extending upwards from that level.

Substituting for ρ from the equation of state gives

$$\frac{dp}{p} = -\frac{g}{R'T_v} dz. \tag{3.2}$$

Integrating this equation, we get

$$p = p_0 \exp \left[-\frac{g}{R'\bar{T}_v} (z - z_0) \right], \tag{3.3}$$

where p is pressure at height z , and \bar{T}_v is the mean virtual temperature

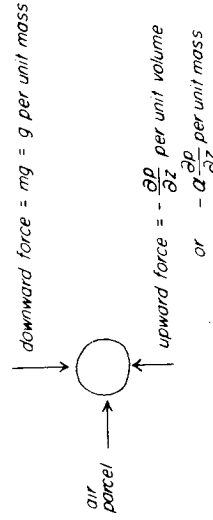


FIG. 3.1. A parcel of air in hydrostatic equilibrium.

Dry adiabatic lapse rate

For dry air undergoing an adiabatic change of pressure,

$$c_p dT = \frac{R'T}{p} dp.$$

Thus for dry air ascending and expanding

$$\frac{dT}{dz} = \frac{R'}{c_p} \frac{T}{p} \frac{dp}{dz}. \tag{3.5}$$

The pressure in an unconfined sample (parcel) of air will immediately adjust to the ambient pressure, so

$$\frac{dp}{dz} = \frac{\partial p}{\partial z} = -\rho'g, \tag{3.6}$$

where ρ' denotes ambient density:

$$\rho' = \frac{p}{R'T'}. \tag{3.7}$$

with T' the ambient temperature.

Combining these equations shows that

$$\frac{dT}{dz} = -\frac{g}{c_p} \frac{T}{T'}.$$

Because the temperature of the parcel is not too different from ambient temperature, $T/T' \approx 1$ and the result simplifies to

$$\frac{dT}{dz} = -\frac{g}{c_p} \equiv -\Gamma, \tag{3.8}$$

where $\Gamma = g/c_p = 0.98^\circ\text{C}/100 \text{ m} \approx 10^{-2} \text{ K/m}$ denotes the *dry adiabatic lapse rate*. This is the rate at which temperature falls off with height in the process of dry adiabatic ascent. It can be shown that the adiabatic lapse rate for moist (but unsaturated) air is equal to Γ to a close approximation.

The assumption that the pressure in an unconfined parcel of air adjusts immediately to the hydrostatic pressure is valid for large-scale vertical air motion but not necessarily for cumulus convection. Parcel theory usually

neglects the pressure deviations from their hydrostatic values, but the dynamic effects of these deviations can be important (Yau, 1979).

Buoyant force on a parcel of air

Consider a parcel of dry air with volume V having temperature T and density ρ . It displaces an equal volume of ambient air having temperature T' and density ρ' . The downward force on the parcel is equal to gV . The downward force on the air displaced is equal to $\rho'gV$. The upward force is the same for parcel and displaced air, $-V(\partial p/\partial z)$. Hence the net buoyant force (upward) is $Vg(\rho' - \rho)$. Therefore, the buoyant force per unit mass is

$$F_B = g \left(\frac{\rho' - \rho}{\rho} \right) = g \left(\frac{T - T'}{T'} \right). \quad (3.9)$$

If this is the only force acting on a parcel, its equation of motion is

$$\frac{d^2z}{dt^2} = F_B = g \left(\frac{T - T'}{T'} \right). \quad (3.10)$$

As expected, this force is positive when the parcel is warmer than ambient air, negative when the parcel is cooler than ambient. For moist air, (3.9) may be generalized by merely replacing the temperatures with virtual temperatures.

Stability criteria for dry air

One of the uses of the dry adiabatic lapse rate is in assessing the stability of atmospheric layers with respect to the vertical displacement of a parcel. If after a small vertical displacement the parcel at a given level is subject to a restoring force which tends to accelerate it toward its original position, the atmosphere at that level is said to be stable. If after displacement the parcel is subject to a force in the direction of the displacement, the atmosphere is said to be unstable. The stability condition depends on the ambient lapse rate, that is, the decrease of temperature with height at the level of the test parcel.

Consider a parcel of air with the ambient temperature T initially. If it is lifted adiabatically a small distance Δz it cools by the amount $\Gamma \Delta z$ and its temperature is reduced to $T - \Gamma \Delta z$. Let us denote the ambient lapse rate by γ , that is,

$$-\left(\frac{\partial T}{\partial z} \right) = \gamma,$$

which is not to be confused with a *process* lapse rate. At height Δz above the initial position of the parcel, the ambient temperature is $T - \gamma \Delta z$. The excess temperature of parcel over ambient air is therefore $\Delta z(\gamma - \Gamma)$.

When this quantity is positive the parcel is warmer than its surroundings and, by (3.9), is accelerated upwards. Consequently the air is unstable whenever $\gamma - \Gamma > 0$. Conversely the parcel is subjected to a restoring force (downward) whenever $\gamma - \Gamma < 0$. For the special case $\gamma = \Gamma$, the displaced parcel experiences zero buoyancy force. The stability criteria for dry air may thus be summarized

$$\begin{aligned} \gamma < \Gamma & \text{ STABLE} \\ \gamma = \Gamma & \text{ NEUTRAL} \\ \gamma > \Gamma & \text{ UNSTABLE} \end{aligned}$$

These criteria may alternatively be expressed in terms of potential temperature. From the defining equation (1.24), the differentials of T , θ , and p are related by

$$dT = \frac{T}{\theta} d\theta + \frac{kT}{p} dp. \quad (3.11)$$

Also, we may take partial derivatives with respect to height and obtain

$$\frac{1}{\theta} \frac{\partial \theta}{\partial z} = \frac{1}{T} \frac{\partial T}{\partial z} - \frac{k}{p} \frac{\partial p}{\partial z} = \frac{1}{T} (\Gamma - \gamma), \quad (3.12)$$

where we have employed (3.1), (3.8), and the equation of state. Equivalently, therefore, the stability conditions may be described by

$$\begin{aligned} \frac{\partial \theta}{\partial z} > 0 & \text{ STABLE} \\ \frac{\partial \theta}{\partial z} = 0 & \text{ NEUTRAL} \\ \frac{\partial \theta}{\partial z} < 0 & \text{ UNSTABLE} \end{aligned}$$

The stability conditions can also be obtained directly from (3.10). By expanding $T(z)$ and $T'(z)$ as Maclaurin's series in z , and neglecting the higher order terms, we find that (3.10) reduces to

$$\frac{d^2z}{dt^2} = -\frac{g}{T} (\Gamma - \gamma)z. \quad (3.13)$$

From (3.12), this result may be written

$$\frac{d^2z}{dt^2} = -\frac{g}{\theta} \left(\frac{\partial \theta}{\partial z} \right) z \equiv -N^2 z, \quad (3.14)$$

where $N = \sqrt{(g/\theta)(\partial \theta/\partial z)}$ has units of s^{-1} and is called the Brunt-Väisälä frequency.

For $N^2 = 0$, the displaced parcel is in neutral equilibrium and there is no restoring force. For $N^2 > 0$, the equilibrium is stable and the parcel undergoes oscillatory motion about its initial point. A typical value of N in the atmosphere is about $1.2 \times 10^{-2} \text{ s}^{-1}$, so that the period of oscillation $\tau = 2\pi/N$ is about 8 min. For $N^2 < 0$, the equilibrium is unstable and the displacement grows exponentially. Obviously, the stability conditions based on N^2 are the same as those given by $\partial\theta/\partial z$.

The pseudoadiabatic lapse rate

Differentiating (2.34) with respect to height gives for the pseudo-adiabatic process

$$\frac{dT}{dz} = \frac{kT}{p} \frac{dp}{dz} - \frac{L}{c_p} \frac{dw_s}{dz} \tag{3.15}$$

By employing the hydrostatic equation and the Clausius-Clapeyron equation, this can be reduced to an expression for the pseudoadiabatic (or saturated adiabatic) lapse rate:

$$\Gamma_s \equiv -\frac{dT}{dz} = \Gamma \left[\frac{1 + \frac{Lw_s}{RT}}{1 + \frac{L^2 \epsilon w_s}{R^2 c_p T^2}} \right] \tag{3.16}$$

It can be seen from (3.16) that $\Gamma_s < \Gamma$ whenever $L\epsilon > c_p T$. Owing to the high value of L for water, this inequality is always satisfied in the atmosphere.

Stability criteria for moist air

When a saturated parcel is displaced upwards its temperature will decrease at the pseudoadiabatic rate. If the ambient lapse rate is greater than pseudoadiabatic, the displaced parcel will find itself warmer than its surroundings and will be accelerated in the direction of the displacement. Such air is unstable with respect to pseudoadiabatic parcel displacement. Allowing for the possibility of condensation on ascent leads to five possible states of stability for moist air:

$\gamma < \Gamma_s$	ABSOLUTELY STABLE
$\gamma = \Gamma_s$	SATURATED NEUTRAL
$\Gamma_s < \gamma < \Gamma$	CONDITIONALLY UNSTABLE
$\gamma = \Gamma$	DRY NEUTRAL
$\gamma > \Gamma$	ABSOLUTELY UNSTABLE

Convective instability

The weight of a column of air with unit cross sectional area which extends from pressure level p_1 up to p_2 is equal to $(p_1 - p_2)$. We consider vertical displacements of this column in which its weight remains constant. (Since g is constant to good approximation, this is equivalent to vertical displacements with mass remaining constant.) Under this condition, $\Delta p = p_1 - p_2$ is constant.

From the hydrostatic equation, the height of the column and its pressure-thickness are related by $\Delta p = g \bar{\rho} \Delta z$ with $\bar{\rho}$ the mean density of air in the column. Since ρ decreases with height, it follows that the lifting process considered here results in changes in Δz : stretching accompanies lifting of the column and contraction accompanies lowering. Usually the stability of the air will be affected by this process.

Consider first the lifting of dry air. Before displacement the stability of the air is measured by $\partial\theta/\partial z$ according to the criteria given previously. Thus over a small height interval δz the potential temperature varies by an amount equal to $\delta\theta = (\partial\theta/\partial z)\delta z$. For this incremental layer, $\delta\theta$ remains constant in adiabatic displacements. However, when the layer is lifted subject to the constraint described δz will increase in consequence of the stretching and it follows that $(\partial\theta/\partial z)$ must decrease. On the other hand $(\partial\theta/\partial z)$ must increase if the column of air is lowered. An exception to this is air having neutral stability, for which $(\partial\theta/\partial z) = 0$ before and after displacement.

These results mean that lifting does not affect the stability of an initially neutral layer. An initially unstable layer becomes less unstable; an initially stable layer becomes less stable. In short, lifting makes the lapse rate tend toward the dry adiabatic. Lowering a layer on the other hand makes its lapse rate depart further from adiabatic.

This effect may be readily illustrated on a thermodynamic chart. Shown in Fig. 3.2 is a layer of 100 mb thickness before and after lifting for the three possible stability conditions. It is noteworthy that air initially stable remains so after being lifted.

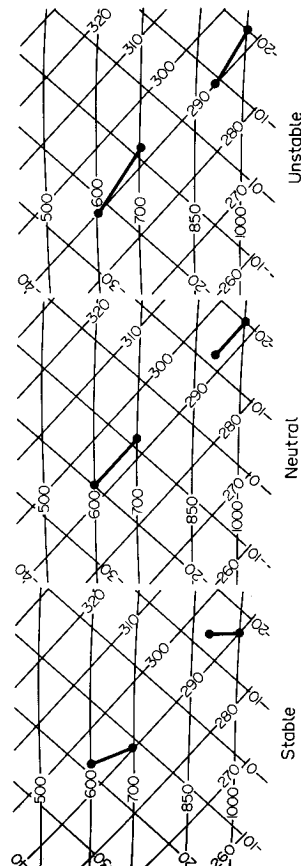


FIG. 3.2. Effect of lifting on stability of a layer of dry air.

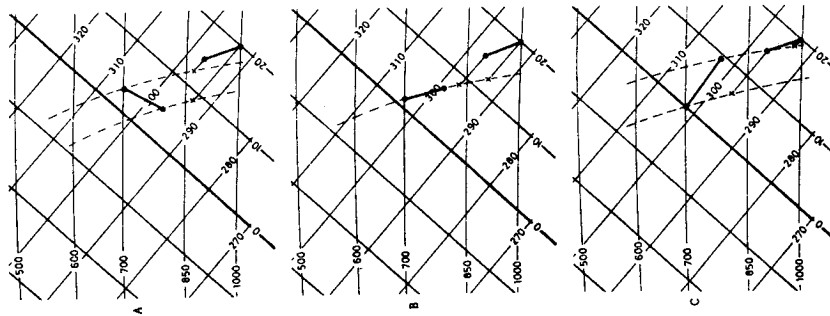


FIG. 3.3. Illustration of criterion for convective instability. In each case the layer between 900 and 1000 mb is lifted to 700 mb and condensation occurs throughout the layer. In case A, $\partial\theta_w/\partial z > 0$ and lifting to condensation stabilizes the layer. Case B has neutral convective stability with $\partial\theta_w/\partial z = 0$. Case C, $\partial\theta_w/\partial z < 0$, is convectively unstable.

Lifting a column of moist air until it is saturated throughout also affects the stability. An important difference between dry and moist air is that moist air, initially stable, may be made absolutely unstable or conditionally unstable by lifting.

A column of air which is rendered unstable by lifting to saturation is said to be convectively unstable. (Some books use the term potentially unstable.) The criteria for convective stability may be expressed in terms of the lapse rate of wet-bulb potential temperature.

$$\frac{\partial\theta_w}{\partial z} > 0 \quad \text{CONVECTIVELY STABLE}$$

$$\frac{\partial\theta_w}{\partial z} = 0 \quad \text{CONVECTIVELY NEUTRAL}$$

$$\frac{\partial\theta_w}{\partial z} < 0 \quad \text{CONVECTIVELY UNSTABLE}$$

These criteria are also best understood with reference to a tephigram (Fig. 3.3).

Convective instability has to do with the lifting of layers and should not be confused with conditional instability, which applies to an undisturbed layer. A layer that is conditionally unstable need not be convectively unstable; nor is a convectively unstable layer necessarily conditionally unstable.

Horizontal restoring forces

So far, our discussion of stability conditions has been limited to the vertical displacement of an air parcel or a layer of air. In the atmosphere, instability can also occur when air is displaced in a slantwise direction. The requirements for slantwise instability can be understood by including horizontal restoring forces in the stability analysis.

The major horizontal forces in the atmosphere are the Coriolis force and the horizontal pressure gradient force. The nature of the Coriolis force can be illustrated by considering an object moving from point C to point B on a turntable rotating counterclockwise as depicted in Fig. 3.4. If the object follows a true straight line in space, the projection of its trajectory on the rotating system will actually be the curve from C to A. To an observer on the rotating system, it would appear that a horizontal force

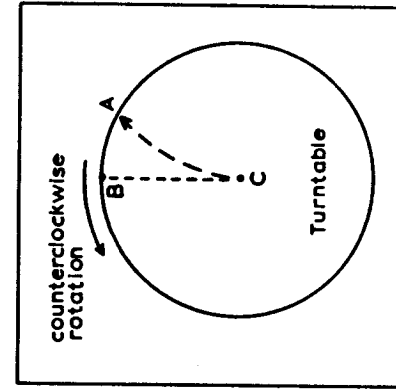


FIG. 3.4. Demonstration of the effect of the Coriolis force in the northern hemisphere. (From Panofsky, 1981.)

has deflected the object toward the right. The magnitude of this force, called the Coriolis force, is equal to twice the product of the local rate of rotation and the speed of the object. Applying this result to the movement of an air parcel in the northern hemisphere, we see that the Coriolis force acts in a direction normal to the trajectory of the parcel and tends to deflect it toward the right. Let x denote the eastward direction and y the northward direction. Then the components of the deflecting force in these directions are given by

$$\left. \begin{aligned} CF_x &= 2\Omega \sin \phi v = fv \\ CF_y &= -2\Omega \sin \phi u = -fu \end{aligned} \right\} \quad (3.17)$$

where Ω is the angular velocity of the earth's rotation (7.29×10^{-5} rad/s), ϕ is the latitude, and u and v are the x and y components of the horizontal wind. $\Omega \sin \phi$ is the local rate of rotation at latitude ϕ . $f = 2\Omega \sin \phi$ is known as the Coriolis parameter.

Geostrophic wind and geostrophic wind shear

The variation of pressure in the horizontal creates horizontal pressure gradient forces, which accelerate an air parcel from high to low pressure. However, as soon as the parcel begins to move, the Coriolis force acts to deflect it toward the right in the northern hemisphere. The magnitude of the force increases as the parcel picks up speed, and eventually a state of balance can be achieved in which the Coriolis force exactly balances the horizontal pressure gradient force. In this state, the air motion and hence the wind is parallel to the isobars. The forces are at right angles to the wind, as shown in Fig. 3.5. The balanced flow, called the geostrophic wind, is described by the relations

$$\left. \begin{aligned} -\frac{1}{\rho} \frac{\partial p}{\partial x} &= -CF_x = -fv_g \\ -\frac{1}{\rho} \frac{\partial p}{\partial y} &= -CF_y = +fu_g \end{aligned} \right\} \quad (3.18)$$

The terms on the left are the pressure gradient forces in the x and y directions. u_g and v_g are the geostrophic wind components.

Because the geostrophic wind is related to the horizontal pressure gradient, any change of the pressure gradient with height implies a vertical variation of the geostrophic wind. This situation often prevails in the atmosphere because of a nonuniform temperature distribution in the horizontal. Figure 3.6 shows a situation in which the geostrophic wind is from the west at sea level, corresponding to increasing surface pressure from north to south. If the air is colder to the north, as shown, the hydrostatic equation then requires that the pressure decrease more

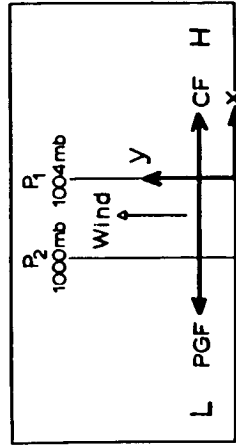


FIG. 3.5. Geostrophic wind in the northern hemisphere, a balance between the pressure gradient force (PGF) and the Coriolis force (CF). (From Panofsky, 1981.)

rapidly with height in the north. The horizontal pressure gradient force at 3 km is therefore stronger than at sea level and the speed of the west wind increases with height.

The variation of the geostrophic wind with height, or the geostrophic wind shear, can be obtained by differentiating (3.18) with respect to z and making use of the hydrostatic equation and the definition of potential temperature. To a good approximation, the result is

$$\left. \begin{aligned} -f \frac{\partial v_g}{\partial z} &= \frac{\partial}{\partial z} \left(-\frac{1}{\rho} \frac{\partial p}{\partial x} \right) = -\frac{g}{\theta} \frac{\partial \theta}{\partial x} \\ f \frac{\partial u_g}{\partial z} &= \frac{\partial}{\partial z} \left(-\frac{1}{\rho} \frac{\partial p}{\partial y} \right) = -\frac{g}{\theta} \frac{\partial \theta}{\partial y} \end{aligned} \right\} \quad (3.19)$$

This shows that the geostrophic wind shear is related to the horizontal gradient of potential temperature. If the potential temperature is uniform in the horizontal, which means that the surfaces on which θ is constant are horizontal, the geostrophic wind does not vary with height. On the other hand, if the surfaces of constant θ are not horizontal, the

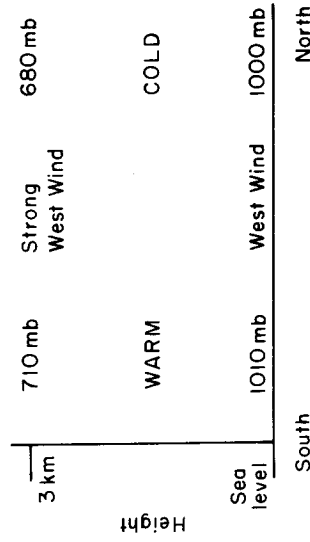


FIG. 3.6. Change of geostrophic wind with height, caused by a horizontal temperature gradient.

geostrophic wind will change with height. Because of its dependence on the horizontal gradient of temperature, the vector difference between the geostrophic wind at two levels is called the thermal wind.

Slantwise displacement

In Fig. 3.7 we consider the situation in which the surfaces of constant potential temperature (the isentropic surfaces) are tilted as indicated. Suppose a parcel of air at point *A* is in equilibrium with the environment. That is, the parcel has the same temperature, potential temperature, pressure, and velocity as its environment. Next we suppose that the parcel is displaced slantwise to position *B*. If condensation does not occur the potential temperature is conserved and the temperature of the parcel at *B* is

$$T + \left(\frac{dT}{dp}\right)dp = T + \frac{kT}{p} \left(\frac{\partial p}{\partial y} \delta y + \frac{\partial p}{\partial z} \delta z\right).$$

The ambient temperature at *B* is given by

$$T + \frac{\partial T}{\partial y} \delta y + \frac{\partial T}{\partial z} \delta z = T + \left(\frac{T}{\theta} \frac{\partial \theta}{\partial y} + \frac{kT}{p} \frac{\partial p}{\partial y}\right) \delta y + \left(\frac{T}{\theta} \frac{\partial \theta}{\partial z} + \frac{kT}{p} \frac{\partial p}{\partial z}\right) \delta z,$$

so that the excess temperature of the displaced parcel over the ambient air is

$$-T \left(\frac{1}{\theta} \frac{\partial \theta}{\partial z} \delta z + \frac{1}{\theta} \frac{\partial \theta}{\partial y} \delta y\right),$$

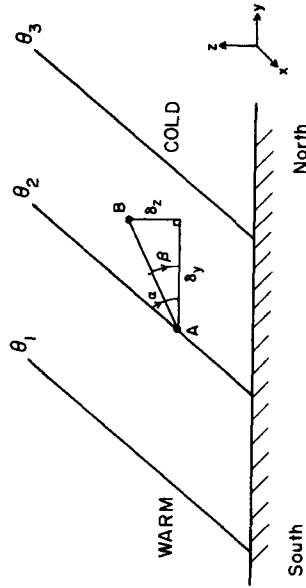


FIG. 3.7. Assessment of parcel stability for slantwise displacement from an equilibrium condition.

and the expression for the buoyancy force on the displaced parcel is

$$F_B = -g \left(\frac{1}{\theta} \frac{\partial \theta}{\partial z} \delta z + \frac{1}{\theta} \frac{\partial \theta}{\partial y} \delta y \right). \tag{3.20}$$

In addition to the buoyancy force, the parcel is subject to a horizontal restoring force because the Coriolis and horizontal pressure forces acting on the parcel are out of balance at the new position. If the parcel moves from *A* to *B* in time δt , the Coriolis force changes the *x*-component of its velocity by an amount $f\nu\delta t = f\delta y$. At its new position, the Coriolis force on the parcel is therefore increased by the amount $f^2\delta y$. The tilt of the isentropic surfaces indicates that the horizontal pressure gradient force must vary with height. From (3.19), the change of the pressure force from *A* to *B* is given by

$$\frac{\partial}{\partial y} \left(-\frac{1}{\rho} \frac{\partial p}{\partial y} \right) \delta y + \frac{\partial}{\partial z} \left(-\frac{1}{\rho} \frac{\partial p}{\partial y} \right) \delta z = f \frac{\partial u_g}{\partial y} \delta y + f \frac{\partial u_g}{\partial z} \delta z.$$

Because the parcel is in equilibrium at *A*, the net horizontal restoring force F_H at *B* is given by the difference between the incremental changes in the Coriolis force and the horizontal pressure gradient force. That is,

$$F_H = f \left[\frac{\partial u_g}{\partial z} \delta z - \left(f - \frac{\partial u_g}{\partial y} \right) \delta y \right]. \tag{3.21}$$

The equation of motion of the parcel along its direction of displacement, with distance denoted by Δ , is therefore

$$\begin{aligned} \frac{d^2\Delta}{dt^2} &= F_B \sin \beta + F_H \cos \beta \\ &= -g \left[\frac{1}{\theta} \frac{\partial \theta}{\partial z} \delta z + \frac{1}{\theta} \frac{\partial \theta}{\partial y} \delta y \right] \sin \beta \\ &\quad + f \left[\frac{\partial u_g}{\partial z} \delta z - \left(f - \frac{\partial u_g}{\partial y} \right) \delta y \right] \cos \beta. \end{aligned} \tag{3.22}$$

This is a generalized equation for parcel displacement, which reduces to (3.14) for vertical displacement by setting $\delta y = 0$ and $\beta = 90^\circ$.

Symmetric instability

A type of slantwise instability, called symmetric instability, can exist for an air parcel that is displaced along an isentropic surface. The buoyancy force then vanishes and the restoring force is given only by F_H . In this situation, the equation analogous to (3.22) becomes

$$\frac{d^2\Delta}{dt^2} = f\delta y \cos \beta \frac{\partial u_g}{\partial z} \left[\frac{\delta z}{\delta y} - \frac{\left(f - \frac{\partial u_g}{\partial y} \right)}{\left(\frac{\partial u_g}{\partial z} \right)} \right] \quad (3.23)$$

If $\partial u_g/\partial z > 0$ and the sign of the factor in square brackets is negative, the displaced parcel will be accelerated towards its initial position, and we describe the initial equilibrium as stable. If this factor is positive, the equilibrium is unstable. The first term in this factor, $\delta z/\delta y$, is just the slope of the isentropic surface. The second term is related to the vorticity of the parcel.

Vorticity is a measure of the local rotational characteristics of a fluid. If the fluid is rotating as a solid body, its vorticity is twice the angular velocity. The vorticity of the atmosphere due to the rotation of the earth equals f . This is called the planetary vorticity and is directed vertically. The absolute vorticity of an air sample is the planetary vorticity plus the vorticity of the air relative to the earth. The relative vorticity of the westerly component of the geostrophic wind, u_g , has two components. Its y component is $\partial u_g/\partial z$ and its z component is $-\partial u_g/\partial y$, so that the z -component of the absolute vorticity is $f - (\partial u_g/\partial y)$. Therefore, in (3.23), the second term in brackets is the ratio of the horizontal to the vertical component of absolute vorticity, and may be interpreted as the slope of the absolute vorticity vector.

The stability conditions for symmetric instability may accordingly be described by

- Slope of isentropic surface
 - < Slope of absolute vorticity vector STABLE
 - = Slope of absolute vorticity vector NEUTRAL
 - > Slope of absolute vorticity vector UNSTABLE

Bennetts and Hoskins (1979) and Emanuel (1979) analyzed this type of instability. They found that in a saturated, cloudy atmosphere, the stability conditions are similar to those for dry air with the exception that the slope of the isentropic surface is replaced by the slope of the surface of wet-bulb potential temperature. Bennetts and Sharp (1982) and Seltzer *et al.* (1985) showed that symmetric instability can be responsible for the mesoscale banded structure of precipitation associated with midlatitude frontal cyclones, as discussed in Chapter 12.

Baroclinic instability

Another type of slantwise instability can occur if only the generalized buoyancy force is included. The equation of motion for the air parcel is then

$$\frac{d^2\Delta}{dt^2} = -g \left(\frac{1}{\theta} \frac{\partial \theta}{\partial z} \right) \delta y \sin \beta \left[\frac{\delta z}{\delta y} - \left(-\frac{\partial \theta}{\partial \theta} \right) / \left(\frac{\partial \theta}{\partial z} \right) \right] \quad (3.24)$$

Because $\delta z/\delta y$ is now the slope of air parcel displacement and $-(\partial\theta/\partial y)/(\partial\theta/\partial z)$ is the slope of the isentropic surface, the stability criteria for a statically stable atmosphere ($\partial\theta/\partial z > 0$) become

- Slope of isentropic surface < Slope of parcel displacement STABLE
- Slope of isentropic surface = Slope of parcel displacement NEUTRAL
- Slope of isentropic surface > Slope of parcel displacement UNSTABLE

Charney (1947) and Eady (1949) were among the first to investigate this so-called baroclinic instability mechanism. The criteria for baroclinic instability are generally met in the atmosphere at midlatitudes and it is firmly established that this kind of instability is responsible for the formation of midlatitude cyclones and the associated widespread cloud and precipitation.

Geopotential

The geopotential $\psi(z)$ is defined as the potential energy of a unit mass at height z above a reference level, usually mean sea level. By this definition, $d\psi = g dz$ is the increase in geopotential for an incremental increase in altitude. Thus there is a close relation between the geopotential at level z and the geometric altitude. As a convenient artifice, the geopotential is measured in units of "geopotential meters", defined by

$$\text{gpm}(z) = \frac{\psi(z)}{g_0} = \frac{1}{g_0} \int_0^z g dz' = \bar{g} z,$$

where g_0 is a standard value of gravity, 9.8 m/s^2 , and \bar{g} is the average value of g between sea level and height z . Meteorological thermodynamic charts such as the tephigram usually have altitude scales, which are derived from the relation between the height in geopotential meters and the pressure in a standard atmosphere. The geopotential height, temperature, pressure, and density in the International Civil Aviation

Organization (ICAO) Standard Atmosphere are given as functions of geometric altitude in the Appendix.

Problems

3.1. Two model atmospheres that are often used in theoretical work are the homogeneous atmosphere, defined by

$$\rho = \text{const.} = \rho_0,$$

and the exponential atmosphere, defined by

$$\rho = \rho(z) = \rho_0 \exp(-z/H),$$

where ρ_0 is the density at the surface and H is called the scale height of the atmosphere. The top of the homogeneous atmosphere is defined as the altitude where its pressure falls to zero. Prove that the height of the top of the homogeneous atmosphere equals the scale height of the exponential atmosphere.

3.2. (a) Prove that the geopotential ψ and the specific enthalpy h of an air sample undergoing a dry adiabatic process are related by $h + \psi = \text{const.}$

(b) Show that the geopotential at pressure level p of an atmosphere in hydrostatic equilibrium is given by

$$\psi(p) = R \bar{T}_p \ln(p_0/p),$$

where $\psi(p_0) = 0$.

3.3. By taking account of the variation of gravity with height, show that the altitude in geopotential meters is related to geometric altitude, to good approximation, by

$$\text{gpm} = z - az^2.$$

Determine the numerical value of a , and solve for gpm at 1, 10, and 50 km.

3.4. In an unstable layer of air over the ground the temperature decreases linearly with height at a rate of $3^\circ\text{C}/100 \text{ m}$. A parcel of air at the bottom of this layer with a temperature of 280 K is given an initial upward velocity of 0.5 m/s . Assuming that the parcel ascends dry adiabatically with no aerodynamic resistance, show that after 1 min it has elapsed it is approximately 45 m above the ground and ascending at 1.28 m/s .

3.5. When deflated, a hot-air balloon, consisting of air bag, gondola, supporting cables, fuel, and burner, has a weight of 1100 N . The balloon has a volume of 1000 m^3 when inflated. Estimate the temperature to which the air in the balloon must be heated for it to float at a steady altitude of 2 km while carrying passengers with a combined weight of 1500 N . Assume a standard atmosphere, and assume that the pressure inside the balloon equals that of the ambient air.

3.6. The boiling point of a liquid is the temperature at which its vapor pressure equals the atmospheric pressure. Derive an expression for the change in the boiling point of water with altitude. Show that for typical sea level conditions the boiling point falls with altitude at a rate of about 3°C per km.

3.7. The mass of water vapor in a vertical column of unit cross sectional area extending from the surface to altitude z is called the columnar vapor. Sometimes called "precipitable water", this quantity has units of kg/m^2 in the SI system or g/cm^2 in the CGS system. Columnar vapor is sometimes defined as the volume equivalent of this mass, in which case the units are $\text{m}^3/\text{m}^2 = \text{m}$ in the SI system or $\text{cm}^3/\text{cm}^2 = \text{cm}$ in CGS.

Derive an expression for the columnar vapor between the surface and altitude z in terms of the mixing ratio. Evaluate this expression assuming that (1) the mixing ratio

decreases exponentially with height with a scale height H_w , and (2) the density decreases exponentially with height with a scale height H . That is,

$$w(z) = w_0 \exp(-z/H_w)$$

$$\rho(z) = \rho_0 \exp(-z/H).$$

For $H = 8 \text{ km}$, $H_w = 4 \text{ km}$, and $\rho_0 = 1 \text{ kg/m}^3$, show that the total columnar water vapor in cm is given in terms of the surface mixing ratio by

$$W = 2.67 \times 10^2 w_0.$$

3.8. Condensation of water can occur in updrafts because the saturation mixing ratio decreases in adiabatic ascent. This property of water can be attributed to the high value of the latent heat of condensation. It has long been speculated that there may be trace gases which, because of low values of L , would condense in downward moving air (Bohren, 1986). Show that the criterion that must be satisfied if vapor is to condense in downdrafts is

$$L < c_p T/\epsilon.$$