Circulation

Review of basics

- Atmosphere is an ideal gas that conserves mass, momentum and energy (thus the primitive equations)
- Horizontal accelerations due to imbalance between pressure gradient and Coriolis
- Vertical balance between pressure gradient and gravity
- Vertical motion leads to expansion work
- For a point in space advection is very important

Review of basics (continued)

- Can simplify the horizontal equations assuming geostrophic balance
- Thermal wind relates horizontal temperature gradient to changes in geostrophic wind with height
- Horizontal divergence associated with vertical motion
- Total divergence in a column changes the mass of the column and so the surface pressure

Vertical motion

- Observations show vertical motions are small compared to horizontal motion (W~0.01 m/s, U ~10m/s)
- Scaling showed that vertical accelerations are tiny (the hydrostatic equation tells us the weight of the air is very close balanced by the vertical pressure gradient)
- Since vertical motion is balanced (hydrostatic), we can not predict changes in vertical velocity by integrating acceleration
- However, we can deduce the vertical velocity diagnostically, since we know vertical motion plays a role in continuity equation (divergence), and thermodynamic equation

Vertical velocity – kinematic method

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

$$d\omega = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp$$

$$\omega(p) = \omega(p_s) + \left(p_s - p\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

We must know the horizontal divergence.

Accurately, if we want accurate vertical velocities

Recall large errors in divergence if small errors in velocity field

This is similar to saying we must be able to estimate the ageostrophic part of the flow, which is the order of the Rossby number smaller than the total flow field

(*Note: vertical velocity also appears in thermodynamic equation, so could use that also… knowing temperature advection and diabatic heating*)

Vertical velocity - thermodynamics

- Atmosphere mostly adiabatic, so thermodynamic equation:

$$\frac{dT}{dt} - \sigma\omega = \frac{J}{c_p} \approx 0$$

$$\sigma = -\frac{T}{\theta}\frac{\partial\theta}{\partial p}$$

$$\omega = \frac{1}{\sigma}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$$

This form is much more robust, because temperature advection estimated from geostrophic assumption and thermal wind balance accurate to the order of the Rossby number.

This form accurate so long as we can assume diabatic heating is small

Pressure tendency

Divergence a measure of air mass movement way from a location

Since pressure is a measure of mass, can use vertical integral of divergence to estimate surface pressure changes

$$d\omega = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dp$$

$$w(p_s) - w(0) = -\int_0^{p_s} (\nabla \cdot \mathbf{V})dp$$

$$w(p_s) = \frac{dp_s}{dt} = \frac{\partial p_s}{\partial t} + V \cdot \nabla p_s$$

$$\frac{\partial p_s}{\partial t} \approx -\int_0^{p_s} (\nabla \cdot \mathbf{V})dp$$

This is a consequence of hydrostatic balance (pressure and mass related).

Divergence and geostrophy

- If the flow is non-divergent in the horizontal, there is no vertical motion
- So ageostrophic conditions lead to vertical motions
- Recall that the acceleration of the flow is due to ageostrophic component of the flow.

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = f v_g \qquad \frac{1}{\rho}\frac{\partial p}{\partial y} = -f u_g$$

$$\frac{du}{dt} = f\left(v - v_g\right) = f v_a \qquad \frac{dv}{dt} = -f\left(u - u_g\right) = -f u_a$$

So can only have vertical motion where there is acceleration (net force imbalance)
This has important implications for development of weather systems

Weekly exercise

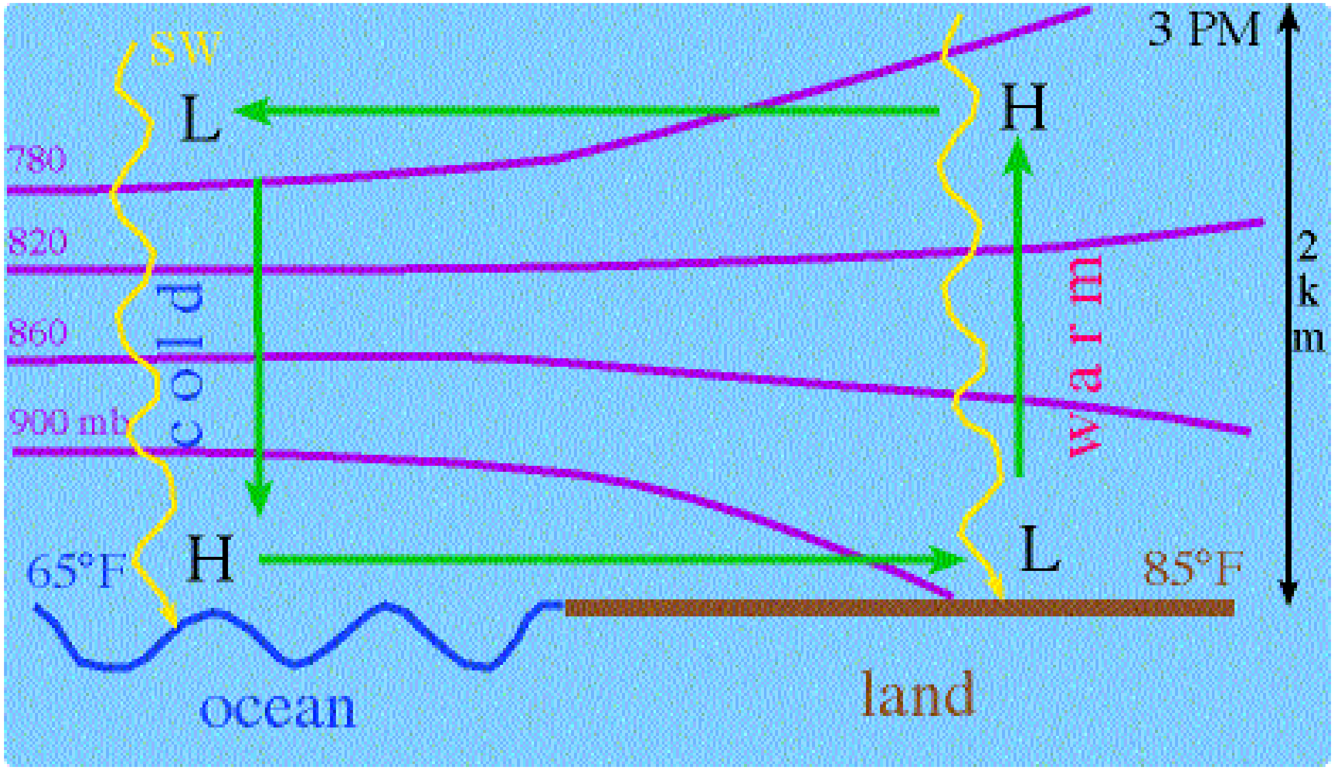
Holton 3.20, 3.21, 3:22

Derive an expression for the divergence of the geostrophic flow, and compute a typical value for large scale midlatitude flow.

Other practice problems:

- Holton: 3.1 – 3.10 are all good review
- Holton: 3.18-3.23

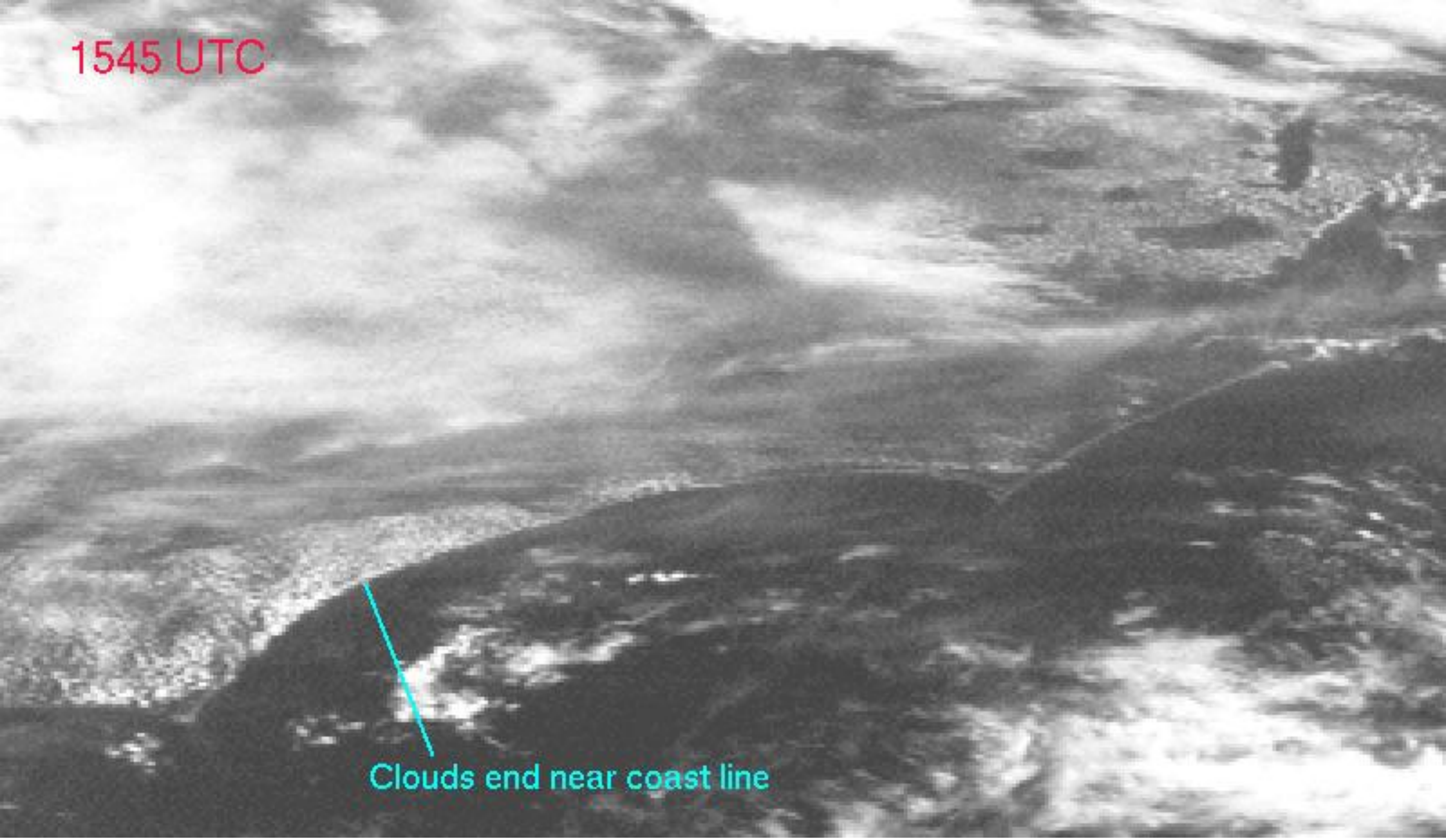
Sea breeze

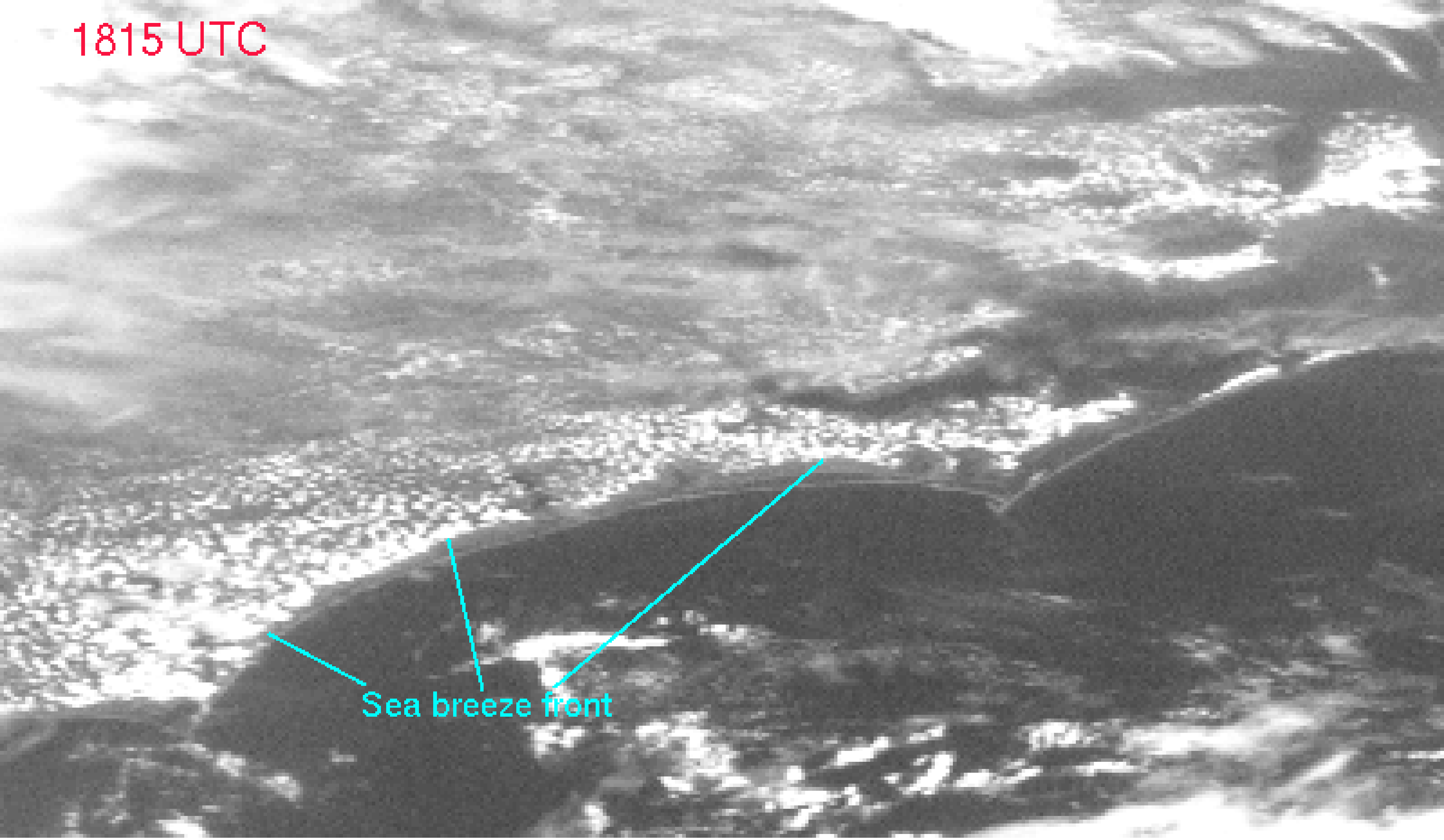


Warmer over land causes ascent, cooler over ocean leads to descent

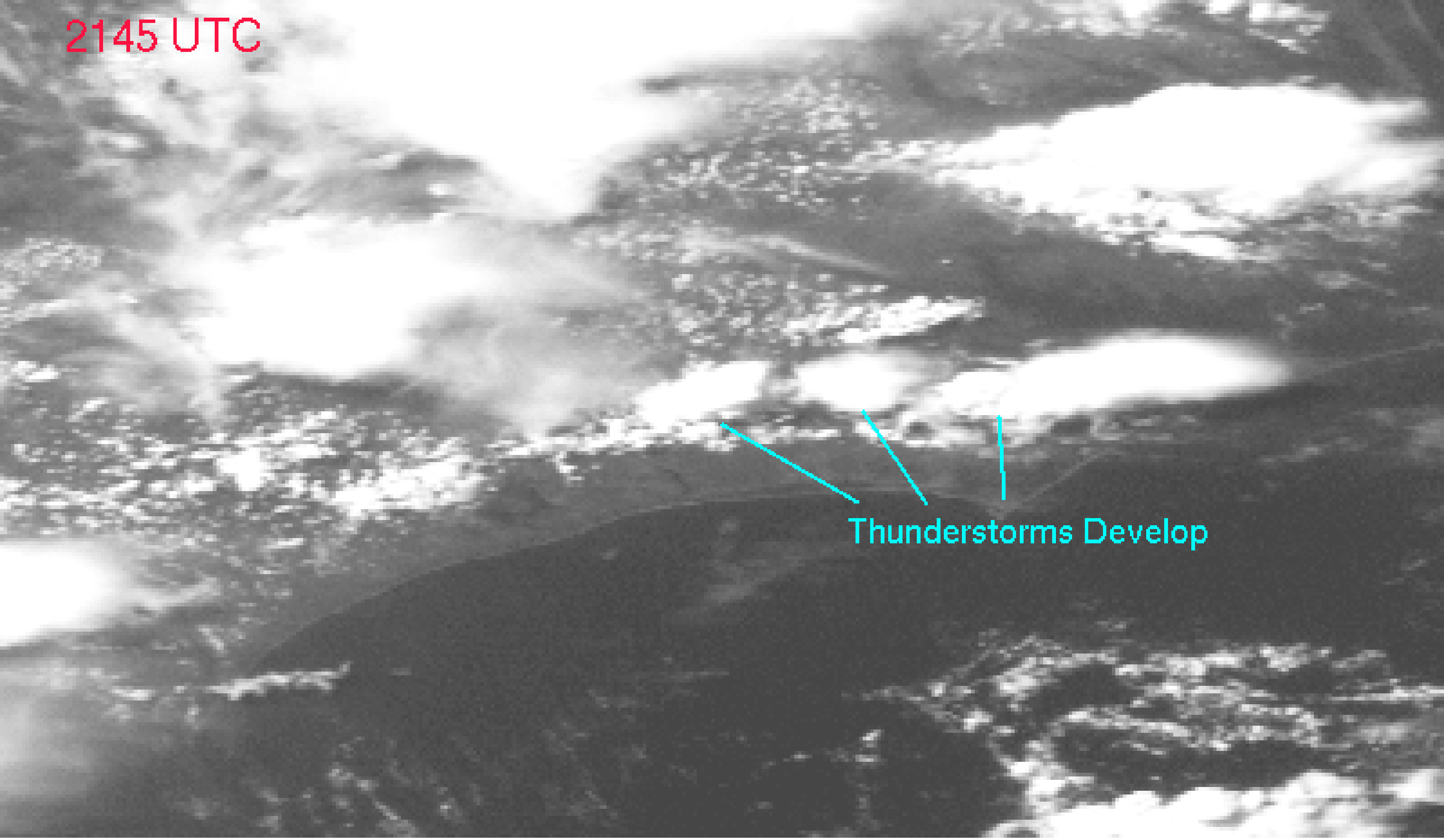
Local temperature changes pressure/density structure

For continuity horizontal and vertical motions linked


1545 UTC
Clouds end near coast line


1815 UTC
Sea breeze front


2015 GMT
Sea breeze penetrates inland


2145 UTC
Thunderstorms Develop

Development of sea breeze *circulation*

Figure 2.

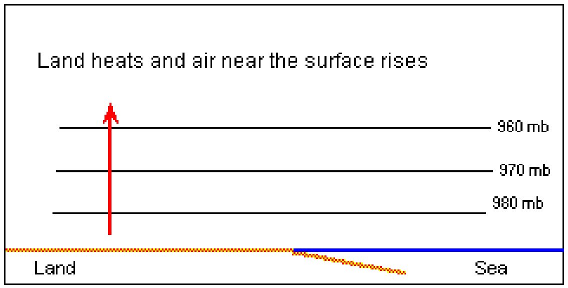


Figure 3.

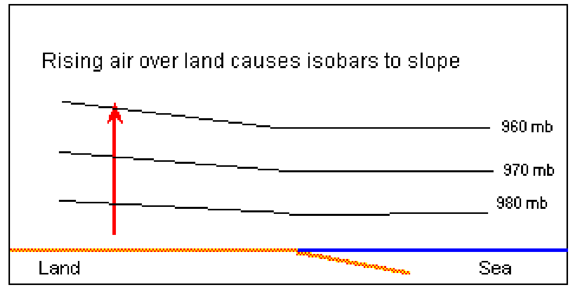


Figure 4.

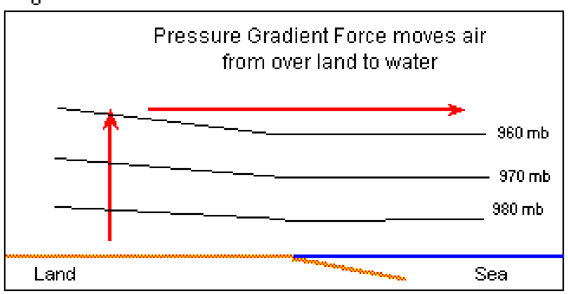


Figure 5.

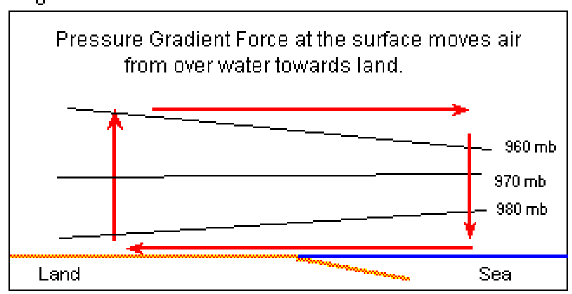
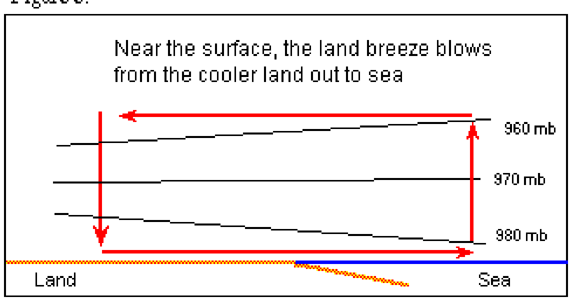


Figure 6.



Fully developed circulation

Need to quantify the “amount” of flow

Also, quantify rate of development
(an acceleration)

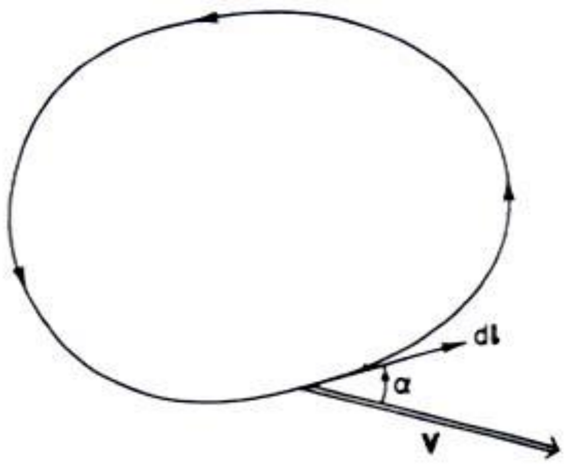
Circulation

- Define a ***property*** that quantifies the amount of rotation for an amount of fluid
- Particular mass of air circulates
- Circulation, C, Defined positive for counter clockwise rotation
  i.e., low pressure cyclones $C > 0$
  high pressure anti-cyclones $C < 0$

  Recall also our natural coordinates, with positive curvature positive as the flow turns cyclonically

Circulation

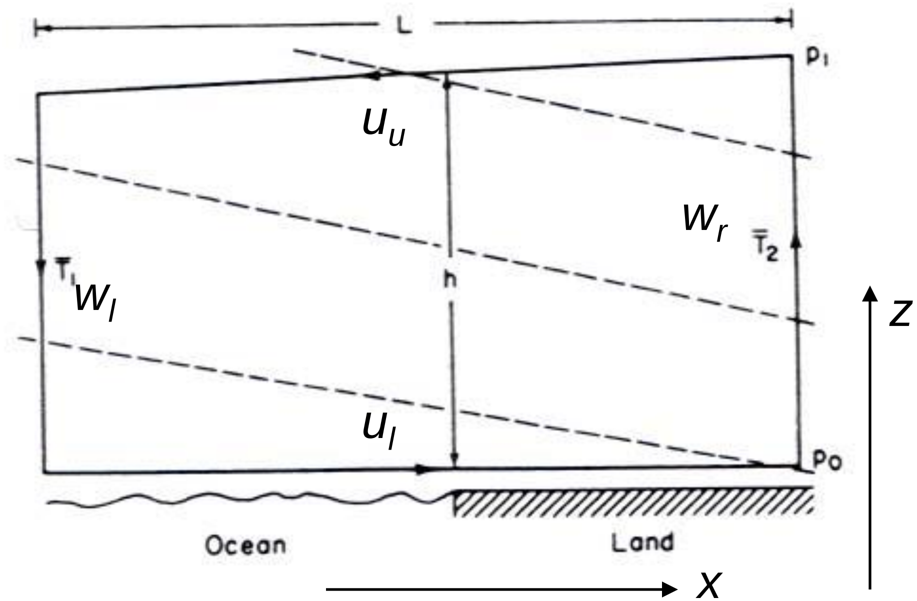
$$C \equiv \oint \mathbf{V} \cdot d\mathbf{l} = \oint |\mathbf{V}| \cos \alpha dl = \oint (u dx + v dy + w dz)$$



- Circulation gives a measure of rotation
- A measure of rotation of fluid enclosed by a ring of parcels
- An analog in fluids for angular momentum in solid bodies
- Can be shown that for solid body rotation C = 2Ω x area
- So, circulation per unit area is twice the rotation rate

Example: sea breeze

What is the circulation?



$$C \equiv \oint \left(u dx + w dz \right)$$

$$C = u_l L + w_r h + u_u L + w_l h$$

Notice value will be different if h or L is different
(i.e., ***circulation depends on area***)

Kelvin circulation theorem

- One the most important principles for fluids
- For fluid flow, the substantive derivative of the line integral of the velocity **V** around a closed path is zero.
- i.e., circulation does not change with time

$$\frac{dC_a}{dt} = 0$$

Strictly, this is true for an “ideal fluid” (thermal conductivity and viscosity are unimportant), for flow at a constant entropy. (i.e., a barotropic atmosphere)

What about fluids with more general conditions?

Circulation theorem

- Compute the change in circulation by examining the acceleration (change in velocity) given by the momentum equation
- For now, in inertial frame, so only gravity and pressure gradient forces

$$\oint \frac{d\mathbf{V}a}{dt} \cdot d\mathbf{l} = -\oint \frac{1}{\rho} \nabla p \cdot d\mathbf{l} - \oint \nabla \Phi \cdot d\mathbf{l}$$

After some work (see Holton),

$$\frac{dC_a}{dt} = -\oint \frac{1}{\rho} dp$$

RHS is "solenoid" term, which exists for baroclinic atmospheres

So circulation conserved for barotropic fluids (RHS = 0)

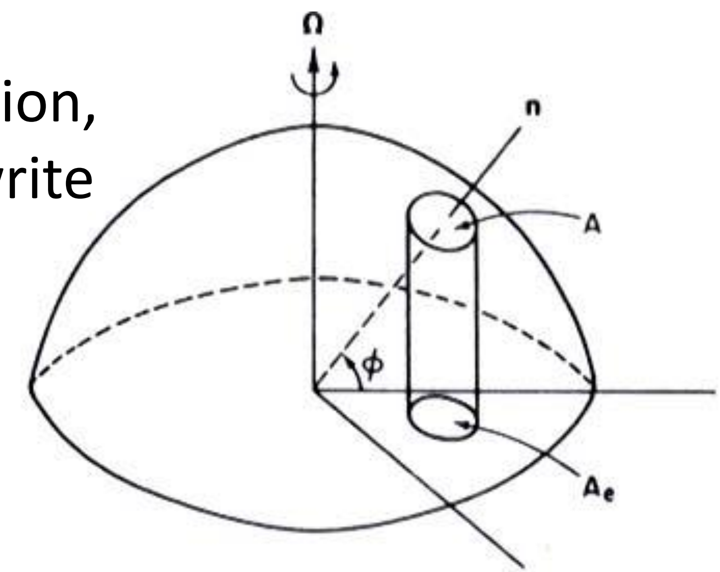
Relative circulation

- Since we are on a rotating sphericical earth it is useful to define “absolute” and “relative” circulation (much as we did for velocity)
- The difference being the local component of the circulation of the Earth

  i.e., relative circulation, $C = C_a - C_e$

- Since the earth is in solid body rotation, solenoid term disappears, and we write

  $C_e = 2\Omega A \sin\phi$



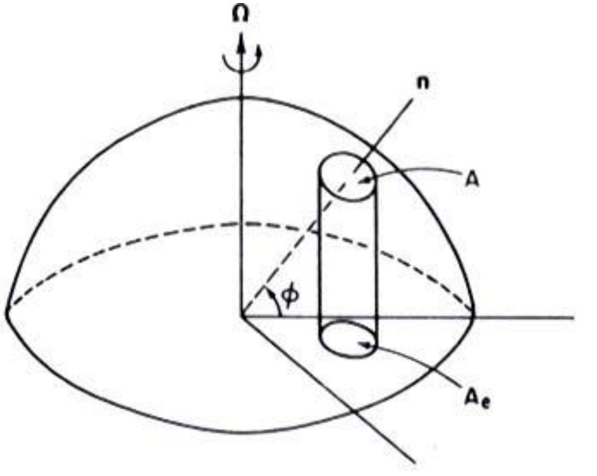
Relative circulation

- $C = C_a - 2\Omega A \sin\phi$
- From definition of circulation (integral of velocity around a ring of parcels)

$$\frac{dC}{dt} = -\oint \frac{1}{\rho} dp - 2\Omega \frac{d}{dt}\left(A \sin\phi\right)$$

For a barotropic fluid, the first term on right drops out.
Thus integrating gives:

$C_2 - C_1 = -2\Omega\ (A_2 \sin\ \phi_2 - A_1 \sin\ \phi_1)$



This is Bjerknes circulation theorem

Example

- A region of air that is motionless (relative to earth) at the equator is moved to the north pole along a pressure surface, while preserving its area.
- Assume a radius of the air mass is 100 km

$C_2 - C_1 = -2\Omega\,(A_2 \sin\phi_2 - A_1 \sin\phi_1)$

$\phi_1 = 0$ and $C_1 = 0$, $\phi_2 = 90$.

so $C = -2\Omega\, A \sin(\phi_2)$

$A = \pi r^2$

Thus we can solve for $C = 4.3 \times 10^6\ m^2/s$

Finally $C = V2\pi r$, so mean tangential $V = -7$ m/s

Historical significance

- “In 1897, … , Vilhelm Bjerknes, discovered the circulation theorem that bears his name. It generalizes Helmholtz's and Kelvin's theorem on vortex conservation in ideal fluids into a theorem on vortex formation in nonhomogeneous fluids. With this theorem, Vilhelm Bjerknes realized that he now was in possession of the complete set of hydrodynamic/thermodynamic equations that govern the motion of nonhomogeneous fluids. Encouraged by his Swedish colleagues, among them the famous chemist Svante Arrhenius [of CO2 fame] and the oceanographer Otto Petterson, he set out to apply the theory to the motions in the atmosphere and the sea. He put forward the view that weather forecasting should be dealt with as an initial value problem of mathematical physics and carried out by numerical or graphical integration of the governing equations. This is nothing more than treating the atmosphere as a physical system; but at the time it was a revolutionary idea.” - Eliassen

Example: sea breeze development

Can use circulation to deduce changes in C.

Here flow is due to density changes

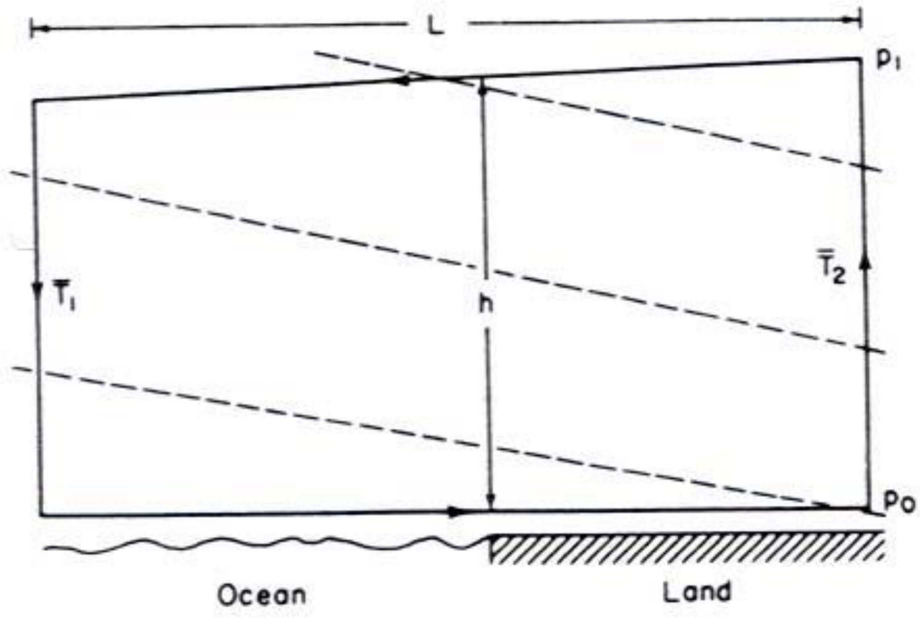
$$\frac{dC_a}{dt} = -\oint \frac{1}{\rho} dp - 2\Omega \frac{d}{dt}(A \sin\phi)$$

$$\frac{dC_a}{dt} = -\oint \frac{1}{\rho} dp$$

$$\frac{dC_a}{dt} = -\oint \frac{RT}{p} dp$$

$$\frac{dC_a}{dt} = -R \ln\left(\frac{p_0}{p_1}\right)(T_2 - T_1)$$

$$\frac{dV}{dt} = \frac{R}{2(h+L)} \ln\left(\frac{p_0}{p_1}\right)(T_2 - T_1)$$



Thus we have “acceleration” of sea breeze due to density changes when heating occurs

Excersize

Holton 4.1, 4.2

Further reading:

- Gill, A. E., 1982: *Atmosphere–Ocean Dynamics*, Academic Press, 226–231 (text with more mathematical detail)
- Thorpe, A. J., Volkert, H., Ziemianski, M., *J.Bjerknes circulation therum – An historical perspective*, Bulletin of the American Meteorological Society, vol. 84, Issue 4, pp.471-480 (PDF on ATOC 5720 web site)