Balanced flow

The primitive equations

- Momentum (x3)

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k}\times\mathbf{V} - \frac{1}{\rho}\nabla p$$

- Continuity

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \bullet \mathbf{V}$$

- Thermodynamic

$$c_p \frac{dT}{dt} - \alpha\frac{dp}{dt} = J$$

(and equation of state)

$$p = \rho RT$$

6 unknowns: *u, v, w, p, T,* ρ

With boundary condition *J* = *J*(*u,v,w,p,T,*ρ*,…*)

Also frictional dissipation.

Isobaric coordinates

- From scaling vertical momentum equation, found hydrostatic balance to be very robust
- Thus define coordinate system *x, y, p* (recall $dx = a \cos\phi \, d\lambda$ and $dy = a \, d\phi$)
- Vertical velocity "omega" $\omega = dp/dt$ (note $\omega < 0$ means up as $dp \sim -dz$)
- Thus expand total (Lagrangian) derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p}$$

Momentum equation

- Prom previous lecture, we showed: (more generally, $(1/\rho)\nabla p = \nabla\Phi$)

$$\frac{1}{\rho}\left(\frac{dp}{dx}\right)_z = \left(\frac{d\Phi}{dx}\right)_p$$

- Thus, (scaled) momentum equation:

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k}\times\mathbf{V} - \nabla\Phi$$

- Density now not present, as coordinate surfaces describe vertical distribution of mass
- *One consequence is horizontal gradients of geopotential at different altitudes have the same geostrophic wind speed (cf. wind speeds is pressure gradient divided by density in height coordinates)*

Thermodynamic equation

Making use of definition of omega:

$$c_p \frac{dT}{dt} - \alpha\omega = J$$

Defining a stability parameter, S:

$$\frac{dT}{dt} - S\omega = \frac{J}{c_p}$$

$$S = -\frac{T}{\theta}\frac{\partial\theta}{\partial p} = \frac{\left(\Gamma_d - \Gamma\right)}{\rho g}$$

Notice the stability increases rapidly with height as density decreases

States that if the atmosphere is very stable, more work is done for the same parcel expansion

Heating directly related to vertical motion

Continuity equation

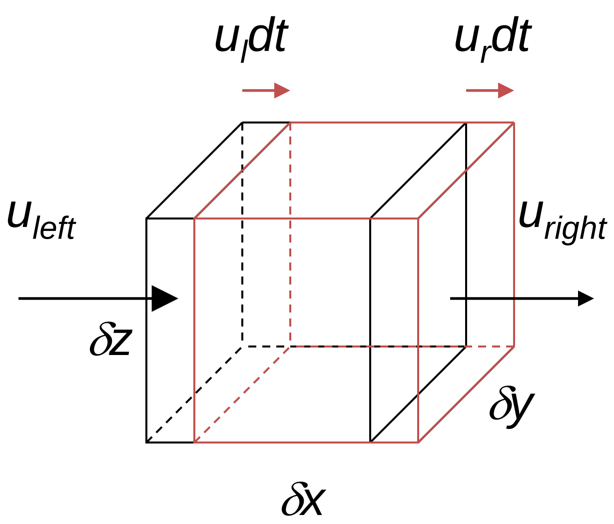
- Consider a Lagrangian parcel (following the motion)
- The mass remains constant, but the volume can change. i.e., $\delta p = -\rho g \delta z$. So $\delta V = -\delta x \delta y \delta p/(\rho g)$ and $\delta m = \delta x \delta y \delta p/g$

$$\frac{d(\delta m)}{dt} = \frac{d(\rho \delta V)}{dt} = \frac{1}{g}\frac{d(\delta x \delta y \delta p)}{dt} = 0$$

$$\frac{1}{\delta x}\frac{d(\delta x)}{dt} + \frac{1}{\delta y}\frac{d(\delta v)}{dt} + \frac{1}{\delta p}\frac{d(\delta p)}{dt} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

$$\nabla \bullet \mathbf{V} = 0$$

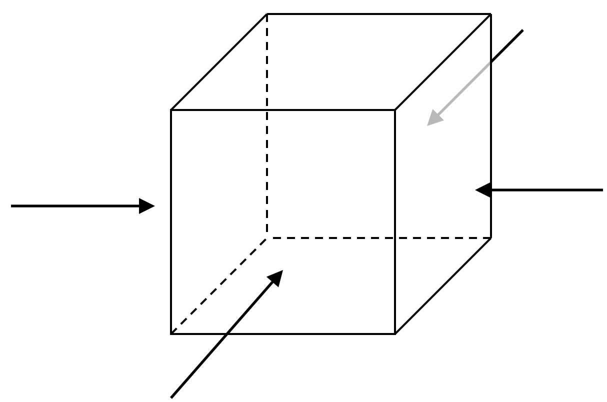


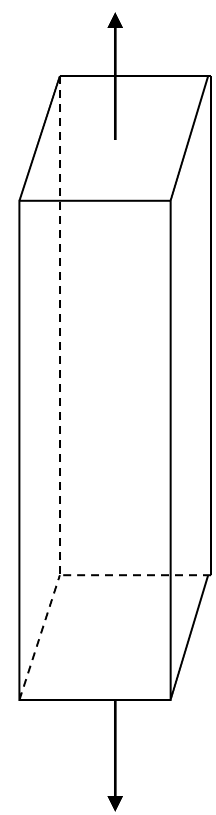
Thus the remarkably simple result. No density!

(compare with *x*, *y*, *z* derivation in from earlier lecture)

Divergence (isobaric)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$$





Primitive equations in isobaric coordinates

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}= fv-\frac{\partial \phi}{\partial x}+\text{friction}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial p}=- fu-\frac{\partial \phi}{\partial y}+\text{friction}$$

$$\frac{\partial \phi}{\partial p}=g$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\omega\frac{\partial T}{\partial p}=\omega\frac{\kappa T}{p}+\frac{J}{c_p}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$$

$$p=\rho RT$$

Notice, additional terms if spherical geometry is used

If we consider a moist atmosphere *T* is the virtual temperature.

These equations have been known for over a century

System has no known analytic solution

Numerical solution is the basis of weather and climate models

Primitive equations are lovely, but…

- They are big and nasty, and we would prefer something more convenient
- Also, there may be some motions which can by explained more simply

Horizontal scaling

Table 2.1 *Scale Analysis of the Horizontal Momentum Equations*

	A	B	C	D	E	F	G
$x-$Eq.	$\frac{Du}{Dt}$	$-2\Omega v \sin\phi$	$+2\Omega w \cos\phi$	$+\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
$y-$Eq.	$\frac{Dv}{Dt}$	$+2\Omega u \sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2\tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
(m s^{-2})	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

Acceleration ~ 10% size of Coriolis and PGF, removing it keep about 90% of the description

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - 2\Omega w\cos\phi + F_{rx}$$

$$\frac{dv}{dt} - \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_{ry}$$

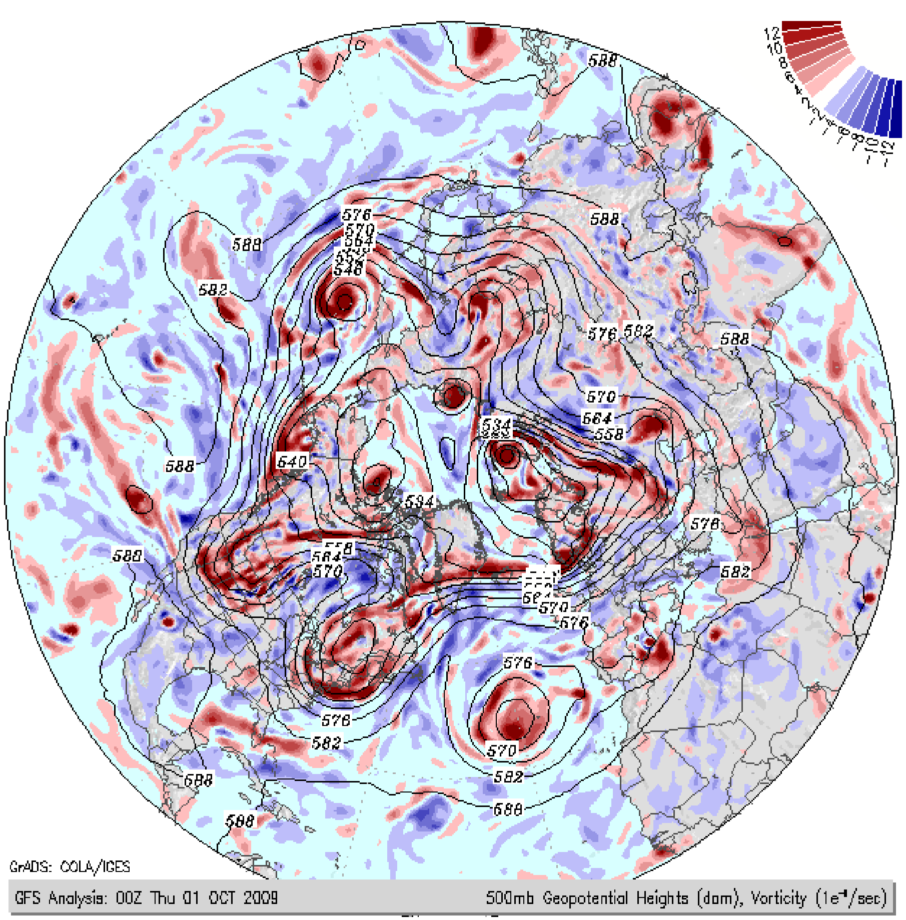
Recall: Geostrophic balance

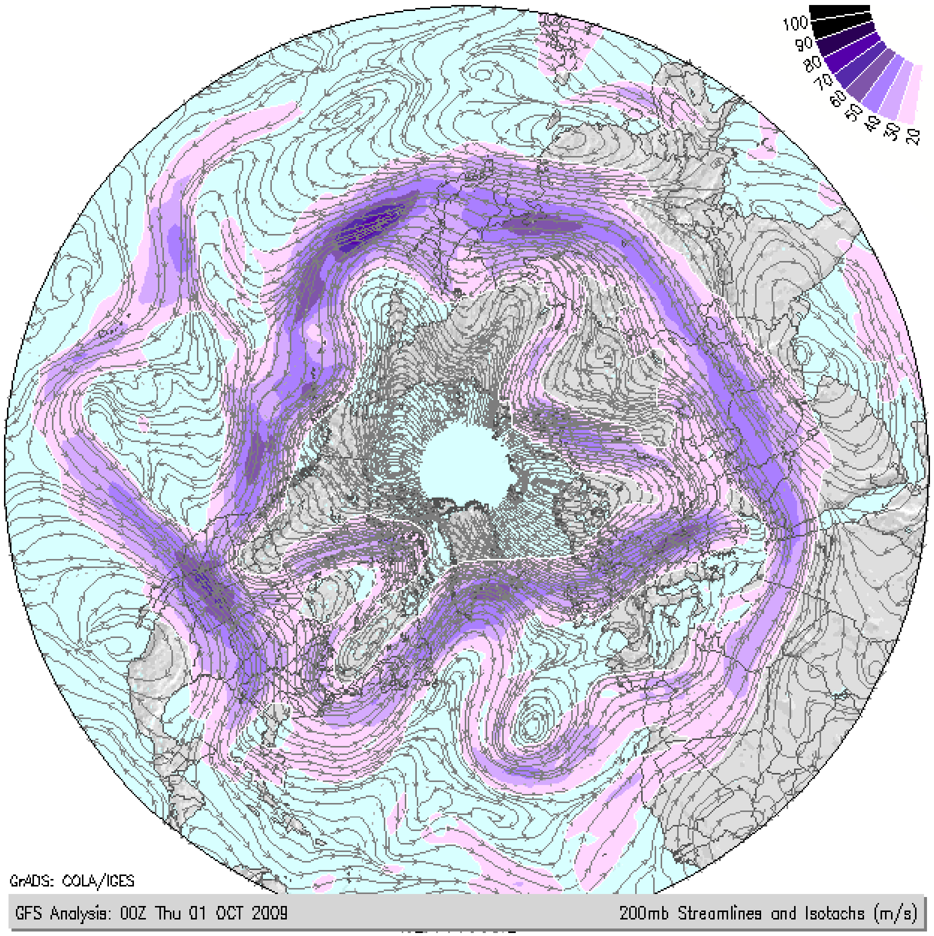
Coriolis and pressure gradient are about the same size

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = fv_g \qquad \frac{1}{\rho}\frac{\partial p}{\partial y} = -fu_g$$

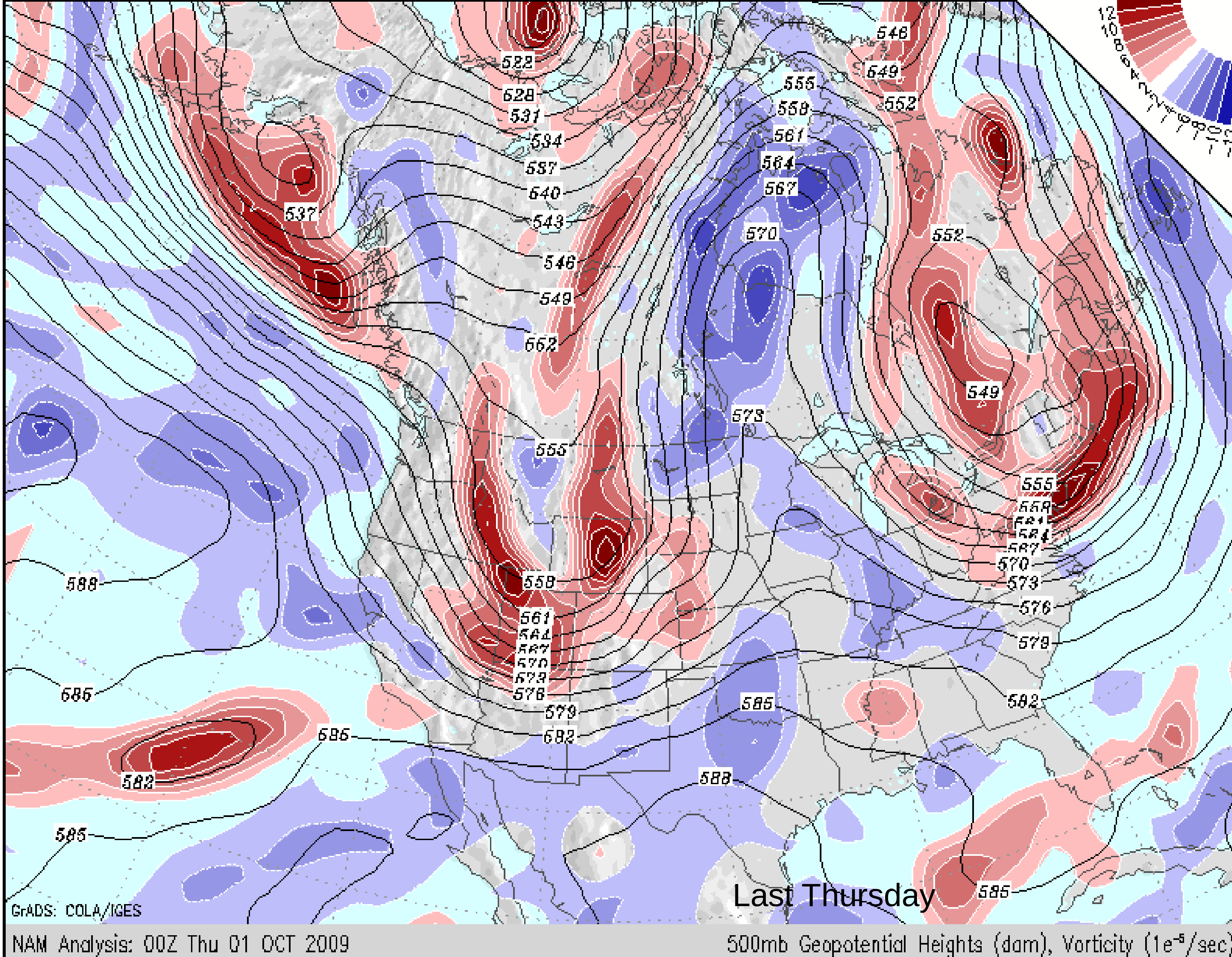
- Very simple version of the description of momentum
- However, remarkably (90%) accurate
- Suggests that flow (*u* and *v*) is aligned perpendicular to the pressure gradient force

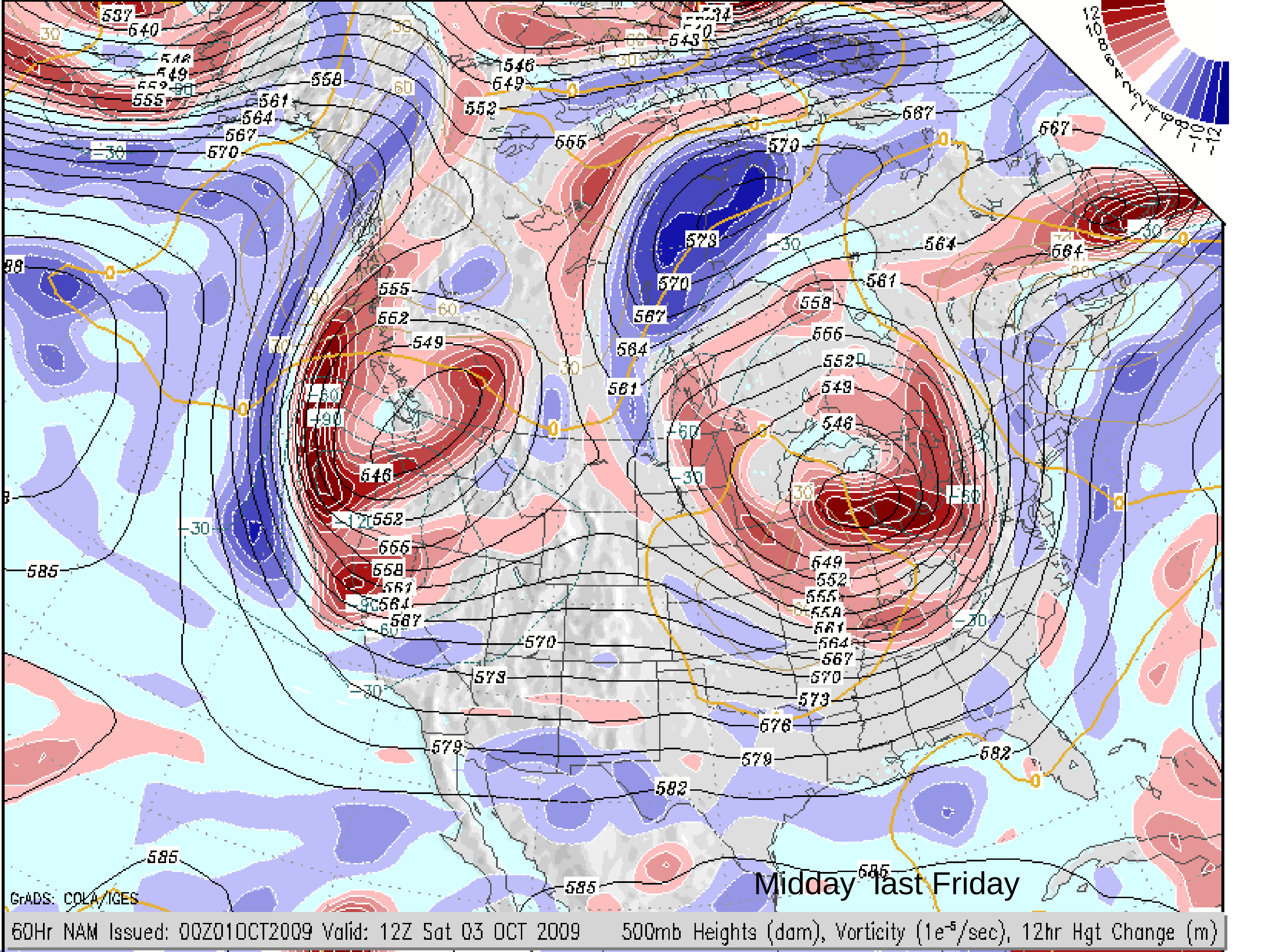
i.e., flow along pressure contours on height surfaces,
or flow along height contours on pressure surfaces





- Geopotential height (contours)
- Wind speed/direction


Last Thursday
GrADS: COLA/IGES
NAM Analysis: 00Z Thu 01 OCT 2009
500mb Geopotential Heights (dam), Vorticity (1e-5/sec)


Midday last Friday
GrADS: COLA/IGES
60Hr NAM Issued: 00Z01OCT2009 Valid: 12Z Sat 03 OCT 2009
500mb Heights (dam), Vorticity (1e-5/sec), 12hr Hgt Change (m)

Natural (intrinsic) coordinates

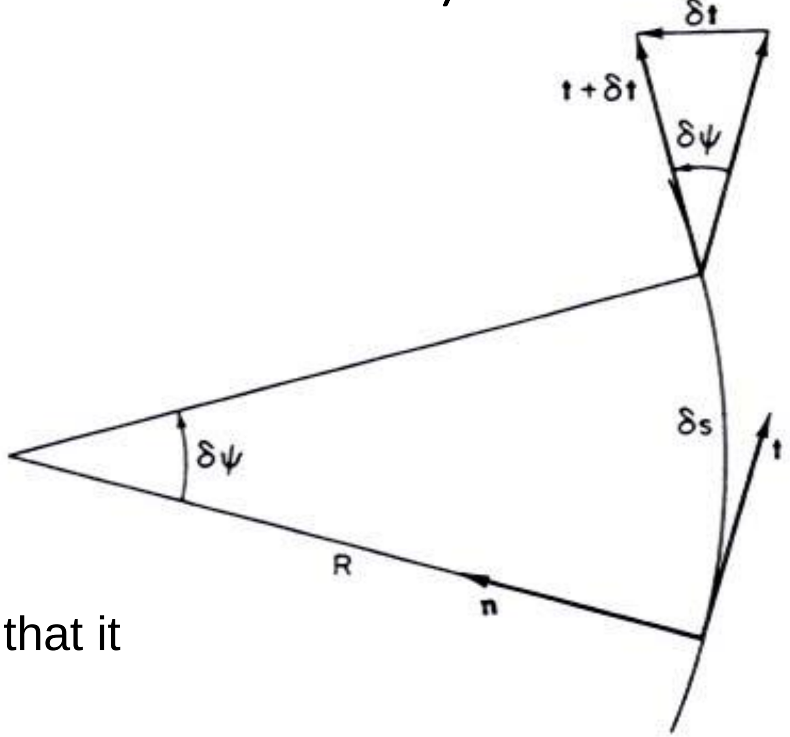
Simplify momentum equations by considering forces in:

1. direction (tangent) of flow, **t**
2. normal to flow, **n**
3. vertical, **k** (but we focus on horizontal)

So,

$\mathbf{V} = V\mathbf{t}$

$V = ds/dt$

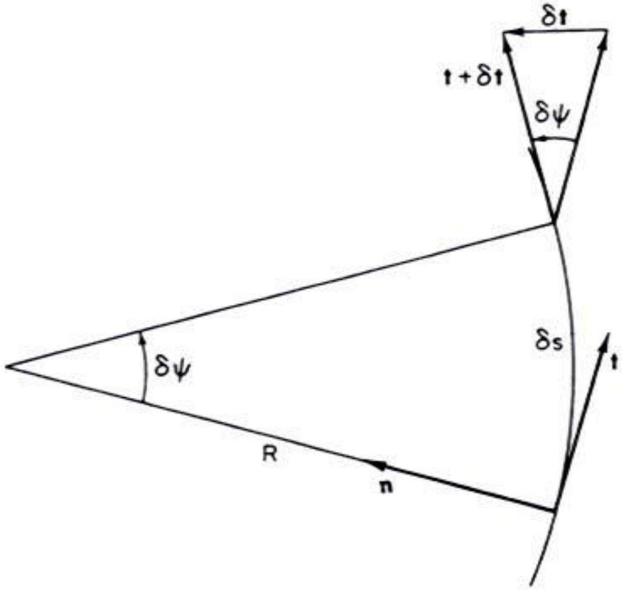


This is a Lagrangian coordinate, in that it changes following the motion

Natural coordinates

Consider change in velocity, and change in coordinate system (as we did for Earth's rotating coordinate)

$$\frac{d\mathbf{V}}{dt} = \frac{dV\mathbf{t}}{dt} = \mathbf{t}\frac{dV}{dt} + V\frac{d\mathbf{t}}{dt}$$



Geometry shows that, $\frac{d\mathbf{t}}{dt} = \mathbf{n}\frac{V}{R}$

$$\frac{d\mathbf{V}}{dt} = \mathbf{t}\frac{dV}{dt} + \mathbf{n}\frac{V^2}{R}$$

change in speed ↗ centripetal acceleration

c.f. $\frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt}$

R is the radius of curvature, and defined positive to the left (i.e., a right-handed coordinate)

Force balance

- Pressure gradient force: $$\nabla\Phi = \mathbf{t}\frac{\partial\Phi}{\partial s} + \mathbf{n}\frac{\partial\Phi}{\partial n}$$

- Coriolis force (normal to flow): $-fV\mathbf{n}$

- Thus tangent and normal components of momentum equation $$\frac{dV}{dt} = -\frac{\partial\Phi}{\partial s} \qquad \frac{V^2}{R} = -fV - \frac{\partial\Phi}{\partial n}$$

Change in speed due to pressure gradient in direction of flow

Thus balanced flow ($d\mathbf{V}/dt$ = 0) must be at constant height
(flow follows contours of geopotential height)

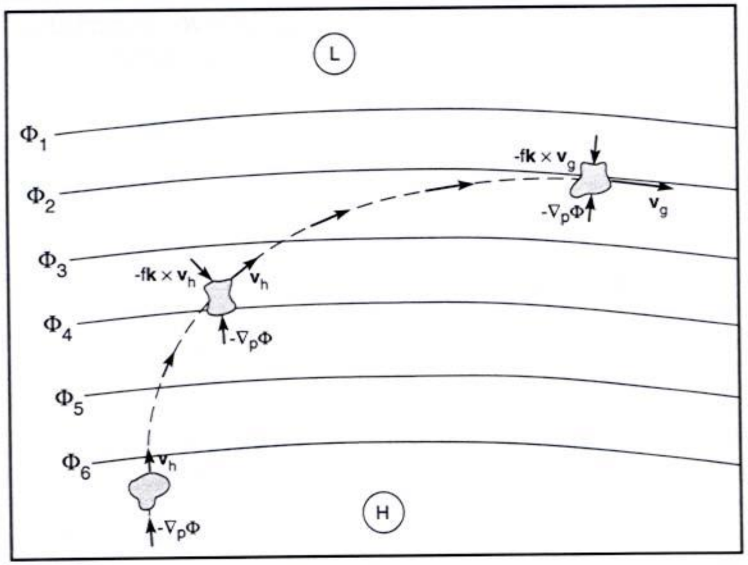
If latitudinal variations in f can be neglected, constant geopotential gradient normal to flow implies constant R
(flow is circular)

Geostrophic flow

- Balance of pressure gradient and Coriolis forces

$$\frac{V^2}{R} = -fV - \frac{\partial \Phi}{\partial n}$$

$$fV = -\frac{\partial \Phi}{\partial n}$$



Strictly, satisfied only as curvature, *R*, becomes infinite ($V^2 << R$)
i.e. flow is straight line

However, a good approximation for most large scale flows
(to the order of the Rossby number)

$$V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$$