Coordinates and scaling

Geopotential

On a non-rotating sphere, gravitational potential energy is constant at the same distance from center of mass.

Masses in the Earth's coordinate frame experience centrifugal force: Define a quantity which is constant for the gravitational force plus centrifugal force.

This is geopotential: Φ

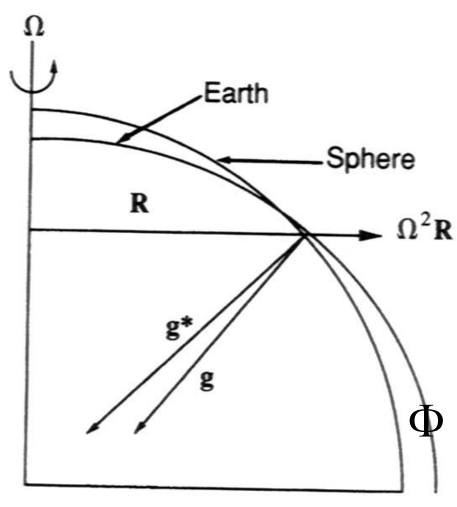
Thus, define *gravity as gradient in geopotential*

$$\mathbf{g} = -\nabla\Phi$$

$$\mathbf{g} = -g\mathbf{k}$$

$$\mathbf{g} \equiv -g^*\mathbf{k} + \Omega^2\mathbf{R}$$

We can interpret this as: "some gravitation must be used to provide centripetal acceleration, the rest is gravity that can be used"



EX 1

Scale analysis

- Consider typical magnitudes of quantities.
- e.g., $g^* \sim 10 m/s^2$, $\Omega \sim 10^{-5}$ /s, $R \sim 10^6$

- So $\Omega^2 R \sim (10^{-5})^2 \times 10^6 = 10^{-4}$
- Thus $g \sim g^*$ to 4 orders of magnitude
- (i.e., very small difference in magnitude, and direction)

EX 2

Momentum equation

In vector form, $$\frac{d\mathbf{V}}{dt} = -2\Omega \times \mathbf{V} - g\mathbf{k} - \frac{1}{\rho}\nabla p + \text{friction}$$

Motion of air (V) relative to rotating Earth view from Earth's rotating reference frame (Ω=0, as per inertial reference frame)

The velocity field, V, is 3 dimensional. We can examine the 3 components separately, but need to define coordinates.

V = *u*i + *v*j + *w*k

u = dx/dt positive to the east (a westerly wind)

v = dy/dt positive to the north (a southerly wind)

w = dz/dt positive upwards

i, j and k are unit vectors pointing east, north and up, and represent an orthogonal coordinate ate system

Spherical coordinates

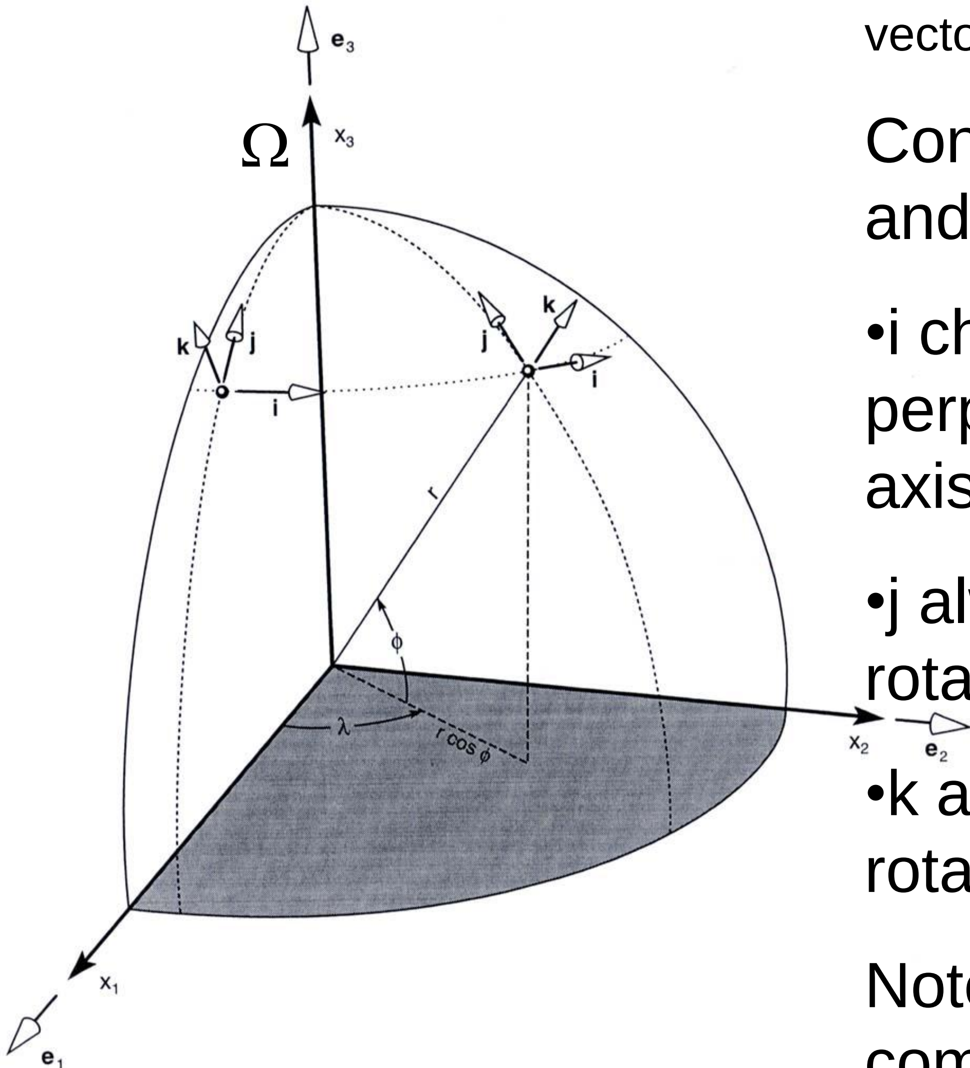
- Earth is very close to spherical (also *r=a+z ~ a*)
- Choose spherical coordinates (λ, ϕ, *z*)
  (longitude and latitude in radians)
- Thus define velocities (change in position)

$$u = r\cos\phi\frac{d\lambda}{dt} \quad v = r\frac{d\phi}{dt} \quad w = \frac{dz}{dt}$$

Use of *r* converts angular measure (radians) to distance (meters)

For *u,* cosϕ take into account distance around a circle of constant latitude (a zone) approaches zero toward the poles
(imagine walking around the "North Pole").
So called, "convergence of the meridians" (lines of constant longitude)

The coordinates move



For coordinates, define unit orthonormal vectors i, j and k

Consider the path of each of i, j and k as Earth rotates

- i changes direction in a plane perpendicular to the rotation axis
- j always points toward the rotation axis
- k always points toward the rotation axis

Note rotating axis Ω is some combination of j and k

$$\Omega = \Omega\,(j \cos \phi + k \sin \phi)$$

Differentiation for rotating frame

- More formally, we need to transfer coordinates from absolute reverence frame to Earths rotating coordinate system
- Define **V** = u**i** + v**j** + w**k**
- Thus derivative of **V**, must account for change in wind and coordinate unit vectors
  (from the chain rule, dAB = AdB + BdA)

$$\frac{d\mathbf{V}}{dt} = \frac{du\mathbf{i}}{dt} + \frac{dv\mathbf{j}}{dt} + \frac{dw\mathbf{k}}{dt}$$

$$\frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt} + u\frac{d\mathbf{i}}{dt} + v\frac{d\mathbf{j}}{dt} + w\frac{d\mathbf{k}}{dt}$$

So we need expressions for changes in unit vectors
We can work through the geometry (see Holton 2.2 for alternate).

Change in vertical coordinate

Using definition k = r/r, and V = dr/dt, use chain rule for the position vector:

$$\frac{d\mathbf{k}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{1}{r}\frac{d\mathbf{r}}{dt} + \frac{\mathbf{r}}{r^2}\frac{dr}{dt}$$

$$\frac{d\mathbf{k}}{dt} = \frac{\mathbf{V}}{r} - \frac{w\mathbf{k}}{r} = \frac{1}{r}\left(u\mathbf{i} + v\mathbf{j} + w\mathbf{k} - w\mathbf{k}\right)$$

$$\frac{d\mathbf{k}}{dt} = \frac{1}{r}(u\mathbf{i} + v\mathbf{j})$$

So k depends only on motion in *x* an *y* directions.

Consider cases *u* = 0, and *v*= 0.
(help to think in terms of angular velocities *u/r* and *v/r*)

Change in northward coordinate

Recall, **$\boldsymbol{\Omega}$ =** Ω (**j** cos ϕ **+ k** sin ϕ*), so* $\hat{\boldsymbol{\Omega}} = \dfrac{\boldsymbol{\Omega}}{\Omega}$

$$\frac{d\hat{\boldsymbol{\Omega}}}{dt} = \cos\phi\frac{d\mathbf{j}}{dt} - \frac{v\sin\phi}{r}\mathbf{j} + \sin\phi\frac{d\mathbf{k}}{dt} + \frac{v\cos\phi}{r}\mathbf{k} = 0$$

Since, we know *v=rdϕ/dt*, and we already have an expression of *dk/dt*, we can simplify this to:

$$\cos\phi\frac{d\mathbf{j}}{dt} = -\frac{v\sin\phi}{r}\mathbf{j} + \frac{\sin\phi}{r}\left(u\mathbf{i} + v\mathbf{j}\right) + \cos\phi\frac{v}{r}\mathbf{k}$$

$$\frac{d\mathbf{j}}{dt} = -\frac{u\tan\phi}{r}\mathbf{i} - \frac{v}{r}\mathbf{k}$$

Change in eastward coordinate

- Since we define an orthogonal coordinate system we know $\mathbf{i}=\mathbf{j}\times\mathbf{k}$, $\mathbf{j}=\mathbf{k}\times\mathbf{i}$ and $\mathbf{k}\times\mathbf{k}=0$.
  Applying the chain rule:

$$\frac{d\mathbf{i}}{dt} = \frac{d\mathbf{j}}{dt}\times\mathbf{k} + \mathbf{j}\times\frac{d\mathbf{k}}{dt}$$

Where the terms on the right are known from before. Substituting and simplifying.

$$\frac{d\mathbf{i}}{dt} = \frac{u\tan\phi}{r}\mathbf{j} - \frac{u}{r}\mathbf{k}$$

Momentum on a sphere

- From sum of forces, we know how the motion changes (gravity, pressure gradient Coriolis)
- We now have expressions for the coordinate vectors

$$\frac{d\mathbf{i}}{dt} = \frac{u\tan\phi}{r}\mathbf{j} - \frac{u}{r}\mathbf{k} \qquad \frac{d\mathbf{j}}{dt} = -\frac{u\tan\phi}{r}\mathbf{i} - \frac{v}{r}\mathbf{k} \qquad \frac{d\mathbf{k}}{dt} = \frac{1}{r}(u\mathbf{i} + v\mathbf{j})$$

So we can directly expand from the transformation equation

We can use the pressure gradient terms directly:

$$\nabla p = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z}$$

Expand the Coriolis term:

$$-2\Omega \times \mathbf{V} = -2\Omega \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos\phi & \sin\phi \\ u & v & w \end{vmatrix}$$

$$-2\Omega \times \mathbf{V} = -2\Omega\left[(w\cos\phi - v\sin\phi)\mathbf{i} + u\sin\phi\mathbf{j} - u\cos\phi\mathbf{k}\right]$$

Momentum equation in spherical coordinates

We have:

- Acceleration ~ Forces (including Coriolis and centrifugal)
- Account of spherical coordinates
- Account of rotating coordinates
- Account of advection

So write down the momentum equation for each of 3 directions

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \frac{uv\tan\phi}{r} + \frac{uw}{r} = -\frac{1}{\rho}\frac{\partial p}{r\cos\phi\partial\lambda} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - \frac{u^2\tan\phi}{r} + \frac{vw}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial\phi} - 2\Omega u\sin\phi + F_{ry}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz}$$

Thankfully, some of these terms are small and can be neglected.

Scale analysis (scaling)

- Compare size of forces (F/m)
- Gravity (we know) g~10m/s , Coriolis f~10^{-4}/s, etc

- Atmosphere:
  L~10^6 m, H~10^4 m, U ~10 m/s, W~10^{-2} m/s, Δp~10^3 Pa, ΔT~10^5s

- e.g., Pressure gradient force
  Horizontal: $(1/\rho)(\partial p/\partial x) = \Delta p/L \sim 10^3/10^6 = 10^{-3}$ m/s^2
  Vertical: $(1/\rho)(\partial p/\partial z) = p_0/H \sim 10^5/10^4 = 10$ m/s^2

- Similarly for all terms in momentum equation

*(Also, could consider the momentum equation for the **ocean:***
L~10^6m, H~10^3m, U~0.1m/s, W~10^{-4} m/s, ΔP 10^3 Pa, $\Delta t \sim 10^7$ s)

Horizontal scaling

Table 2.1 *Scale Analysis of the Horizontal Momentum Equations*

	A	B	C	D	E	F	G
$x-$Eq.	$\frac{Du}{Dt}$	$-2\Omega v\sin\phi$	$+2\Omega w\cos\phi$	$+\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
$y-$Eq.	$\frac{Dv}{Dt}$	$+2\Omega u\sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	f_0U	f_0W	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
$(\mathrm{m\ s^{-2}})$	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

- Notice close balance between Coriolis and pressure gradient (acceleration only 90% smaller). *So called geostrophic balance.*

$$\frac{du}{dt}-\frac{uv\tan\phi}{a}+\frac{uw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\Omega v\sin\phi-2\Omega w\cos\phi+F_{rx}$$

$$\frac{dv}{dt}-\frac{u^2\tan\phi}{a}+\frac{vw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\phi+F_{ry}$$

Notice we have also assumed the atmosphere is shallow: r ~ a

Geostrophic balance

Coriolis and pressure gradient are about the same size

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = fv_g \qquad \frac{1}{\rho}\frac{\partial p}{\partial y} = -fu_g$$

•Very simple version of the description of momentum

•However, remarkably accurate

•Suggests that flow (*u* and *v*) is aligned perpendicular to the pressure gradient force
i.e., flow along pressure contours on height surfaces,
or flow along height contours on pressure surfaces

•Notice there is no acceleration term, so we can not predict the motion with the geostrophic equations

•Useful for "diagnostic" purposes only

Vertical scaling

Table 2.2 *Scale Analysis of the Vertical Momentum Equation*

z - Eq.	Dw/Dt	$-2\Omega u \cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	$f_0 U$	U^2/a	$P_0/(\rho H)$	g	$\nu W H^{-2}$
m s^{-2}	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

Notice pressure gradient and gravity *much* larger

$$\frac{dw}{dt} - \frac{u^2+v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz}$$

- Hydrostatic assumption is very robust.
- What are scales needed for non-hydrostatic conditions (L~ 10km… thunderstorms, etc)

Filtered momentum equation

- Thus, the simplified set of equations:

$$\frac{\partial u}{\partial t} = -\mathbf{V} \bullet \nabla u - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} = -\mathbf{V} \bullet \nabla v - \frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

$$\frac{\partial p}{\partial z} = -\rho g$$

* Note, for some global problems also include some of the *uv* and u^2 “metric” terms

In vector form of horizontal terms: $\frac{d\mathbf{V}}{dt} + f\mathbf{k} \times \mathbf{V} = -\frac{1}{\rho}\nabla p$

Now that we have build this equation from solid principles, as can use it as a starting point for further analysis

Momentum in the ocean

- Applying scaling arguments for the ocean
  (in this case we can not neglect viscosity)

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial z}$$

Like the atmosphere, ocean has a dominant geostrophic balance

Consider forces acting *vertically*

- Forces for parcel at rest?
- Gravity balanced with pressure gradient

$$0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

The hydrostatic equation

This is a very robust approximation for the large-scale atmosphere

Pressure is mass

- Pressure defined as force (imparted on parcel walls by molecules) per units area

$$p = F/A$$

- For horizontal surface the force is weight

$$F = mg$$

- So, mass:

$$m = F/g = pA/g$$

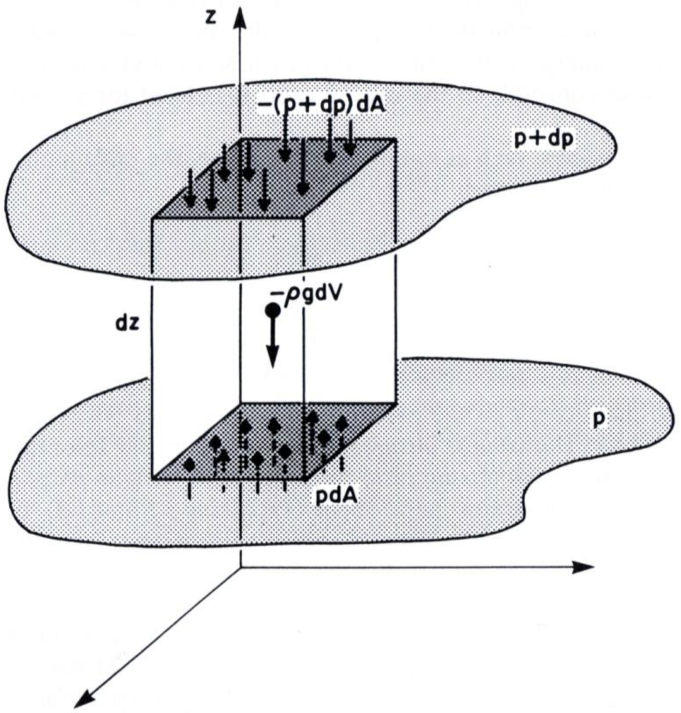
Pressure and mass

- Compute pressure at some height by integrating:

$$p(z) = \int_0^p \mathrm{d}p = -g\int_\infty^z \rho \mathrm{dz}$$

- Compute sea level pressure

$$p_s = -g\int_\infty^0 \rho \mathrm{dz}$$



If the surface pressure is 1000hPa, what is the pressure with half the atmospheric mass above?

Half! (500hPa)

Hypsometric equation

- To integrate, realize density depends on pressure (ideal gas law $p=\rho RT$), and recalling $g = \partial\Phi/\partial z$

$$g\int_{z_2}^{z_1} \mathrm{d}z = -\int_{p_2}^{p_1} \frac{RT}{p}\,\mathrm{d}p$$

$$\Phi(z_2) - \Phi(z_1) = R\int_{p_2}^{p_1} T\mathrm{d}\ln p$$

Define geopotential height, $Z = \Phi(z)/g_0$

Since $g \sim g_0$, $Z \sim z$ for meteorological applications

(The difference being, we have cleverly avoided terms for the centrifugal force)

Thickness and temperature

$$Z_{thk} = Z_2 - Z_1 = \frac{R}{g_0} \int_{p_2}^{p_1} T \mathrm{d} \ln p$$

Assuming a (mass weighted) mean temperature,

$$Z_{thk} = H \ln\left(\frac{p_1}{p_2}\right)$$

$$H = \frac{R\overline{T}}{g_0}$$

H is the “scale height”

Conclude that the thickness between isobaric layers is an approximate measure of the mean temperature

Exercises

For Tuesday

- Holton 1.16 (good one for tropospheric modelers)
- Holton 1.18 (good for middle atmosphere modelers)

For Thursday

- Holton2.6
- Holton 2.7